

# Bäcklund Transformations for Darboux Integrable Differential Systems

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# 1 Introduction

Bäcklund transformations have a long and distinguished history in the study of integrable differential equations. From the geometric viewpoint of exterior differential systems theory, two differential systems  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , defined on manifolds  $M_1$  and  $M_2$ , are said to be related by a Bäcklund transformation if there exists a differential system  $\mathcal{B}$  on a manifold  $N$  and maps

$$\begin{array}{ccc} & (\mathcal{B}, N) & \\ \mathbf{p}_1 \swarrow & & \searrow \mathbf{p}_2 \\ (\mathcal{I}_1, M_1) & & (\mathcal{I}_2, M_2) \end{array} \quad (1.1)$$

which define  $\mathcal{B}$  as integrable extensions for both  $\mathcal{I}_1$  and  $\mathcal{I}_2$ . Integral manifolds of  $\mathcal{I}_1$  can then be lifted, by solving ordinary differential equations, to integral manifolds of  $\mathcal{B}$  which then project by  $\mathbf{p}_2$  to integral manifolds of  $\mathcal{I}_2$  and conversely.

The purpose of this paper is two-fold:

- [i] to show that Bäcklund transformations can easily be constructed using the general theory of symmetry reduction of exterior differential systems; and
- [ii] to prove that this symmetry approach to Bäcklund transformations leads to a very general, yet precise, understanding of Bäcklund transformations for Darboux integrable differential systems.

Our first main result describes a very simple, group-theoretic method for constructing Bäcklund transformation. Let  $\mathcal{I}$  be a differential system on manifold  $M$  and let  $\mathbf{p} : M \rightarrow N$  be a smooth submersion. We define the *reduced differential system*  $\mathcal{I}/\mathbf{p}$  on  $N$  by

$$\mathcal{I}/\mathbf{p} = \{ \theta \in \Omega^*(N) \mid \mathbf{p}^*(\theta) \in \mathcal{I} \}.$$

In the special case where  $G$  is a symmetry group of  $\mathcal{I}$  which acts regularly on  $M$ , we shall write  $\mathcal{I}/G$  in place of  $\mathcal{I}/\mathbf{q}_G$ , where  $\mathbf{q}_G : M \rightarrow M/G$  is the canonical projection to the space of orbits  $M/G$ .

**Theorem A.** *Let  $\mathcal{I}$  be a differential system on  $M$  with symmetry groups  $G_1$  and  $G_2$ . Let  $H$  be a common subgroup of  $G_1$  and  $G_2$  and assume that*

- [i]  *$M/H$ ,  $M/G_1$  and  $M/G_2$  are smooth manifolds with smooth quotient maps  $\mathbf{q}_H : M \rightarrow M/H$ ,  $\mathbf{q}_{G_1} : M \rightarrow M/G_1$ , and  $\mathbf{q}_{G_2} : M \rightarrow M/G_2$ .*

*Then*

$$\begin{array}{ccccc} & & \mathcal{I} & & \\ & \swarrow & \downarrow & \searrow & \\ & \mathbf{q}_{G_1} & \mathbf{q}_H & \mathbf{q}_{G_2} & \\ & & \mathcal{I}/H & & \\ & \swarrow & & \searrow & \\ \mathcal{I}/G_1 & & & & \mathcal{I}/G_2 \\ & \nwarrow \mathbf{p}_1 & & \nearrow \mathbf{p}_2 & \end{array} \quad (1.2)$$

is a commutative diagram of EDS. Furthermore, if

[ii] the actions of  $G_1$  and  $G_2$  are transverse to  $\mathcal{I}$ ,

then the maps  $\mathbf{q}_{G_1}$ ,  $\mathbf{q}_{G_2}$ ,  $\mathbf{p}_1$  and  $\mathbf{p}_2$  all define integrable extensions. The differential system  $\mathcal{B} = \mathcal{I}/H$ , together with the orbit projection maps  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , defines a Bäcklund transformation between  $\mathcal{I}_1 = \mathcal{I}/G_1$  and  $\mathcal{I}_2 = \mathcal{I}/G_2$ .

Note that the fiber dimensions for the orbit projection maps  $\mathbf{p}_1$  and  $\mathbf{p}_2$  defining the Bäcklund transformation (1.2) are given by the difference of orbit dimensions  $\dim \mathcal{O}_{G_1} - \dim \mathcal{O}_H$  and  $\dim \mathcal{O}_{G_2} - \dim \mathcal{O}_H$ , respectively. Thus, the in particular case of free actions, Theorem A can be used to construct a Bäcklund transformation with 1-dimensional fibers only when  $G_1$  and  $G_2$  have a common subgroup of co-dimension 1.

Theorem A provides a powerful method for constructing Bäcklund transformations for a differential system  $\mathcal{I}_2$  for which a non-trivial realization as a quotient differential system  $\mathcal{I}_2 = \mathcal{I}/G$  is known. In a recent article [3] the authors gave a significant generalization of the notion of Darboux integrability and proved that Darboux integrable systems always admit such a quotient representation. The Lie group  $G$  used to construct the quotient representation is called the **Vessiot group** of the Darboux integrable system, in recognition of E. Vessiot's pioneering contributions to the study of such equations [35], [36]. Our next two theorems, Theorems B and C, demonstrate the decisive role that the Vessiot group plays in the construction of Bäcklund transformations for Darboux integrable systems.

**Theorem B.** *Let  $\mathcal{I}_2$  be a Darboux integrable differential system with Vessiot group  $G_2$  and quotient representation  $\mathcal{I}_2 = \mathcal{I}/G_2$ . Assume that  $\dim G_2 > 1$ . Then there always exists a second symmetry group  $G_1$  of  $\mathcal{I}$  for which the commutative diagram (1.2) can be used to construct a Bäcklund transform between  $\mathcal{I}_2$  and the differential system  $\mathcal{I}_1 = \mathcal{I}/G_1$ . The differential systems  $\mathcal{B}$  and  $\mathcal{I}_1$  are Darboux integrable and  $\mathcal{I}_1$  has more (functionally independent) Darboux invariants than  $\mathcal{I}_2$ .*

In this way one can construct a chain of Bäcklund transformations for Darboux integrable differential system with increasing numbers of Darboux invariants. The chain terminates at a Darboux integrable system whose Vessiot group has dimension  $\leq 1$ . Theorem B allows us to quickly determine the Bäcklund transformations for all the Darboux integrable Monge-Ampère equations of the type considered in [13], [14] and [38]. In addition, we shall use Theorem B to construct new Bäcklund transformations for equations not of Monge-Ampère type; for equations which are Darboux integrable at higher jet levels; for systems of equations in several dependent variables; and for over-determined systems in 3 independent variables.

The second main result of the paper establishes a partial converse to Theorem B, that is, given a Bäcklund transformation  $\mathcal{B}$  for a Darboux integrable system  $\mathcal{I}_2$ , we prove that  $\mathcal{B}$  itself is always Darboux integrable and we give necessary and sufficient conditions (which we call **Darboux compatibility**) under which the right-hand side of the Bäcklund transformation (1.1) coincides with the

bottom right-hand side of (1.2). We formulate this converse as a theorem on integrable extensions of Darboux integrable systems.

**Theorem C.** *Let  $(\mathcal{I}_2, M_2)$  be a Darboux integrable differential system with Vessiot group  $G_2$  and quotient representation  $(\mathcal{I}_2, M_2) = (\mathcal{I}/G_2, M/G_2)$ .*

- [i] *If  $\mathbf{p} : (\mathcal{E}, N) \rightarrow (\mathcal{I}_2, M_2)$  is an integrable extension, then  $\mathcal{E}$  is Darboux integrable.*
- [ii] *Futhermore, if the pair of systems  $\mathcal{E}$  and  $\mathcal{I}_2$  are Darboux compatible, then the Vessiot group  $H$  of  $\mathcal{E}$  is a subgroup of  $G_2$  and the differential systems  $(\mathcal{E}, N)$  and  $(\mathcal{I}/H, M/H)$  are locally equivalent.*

If  $\mathcal{B}$  is a Bäcklund transformation between the wave equation  $v_{xy} = 0$  (defining the system  $\mathcal{I}_1$ ) and a scalar Darboux integrable equation  $u_{xy} = f(x, y, u, u_x, u_y)$  (defining the system  $\mathcal{I}_2$ ) then, under the hypothesis considered in [14], we find that the pair  $\mathcal{B}, \mathcal{I}_2$  (or more precisely, the pair of first prolongations  $\mathcal{B}^{[1]}, \mathcal{I}_2^{[1]}$ ) is always Darboux compatible. Theorem C then implies that the Vessiot group  $H$  of  $\mathcal{B}$  must be a subgroup of the Vessiot group  $G$  of  $\mathcal{I}_2$ . Consider, for example, the equation

$$u_{xy} = \frac{\sqrt{1 - u_x^2} \sqrt{1 - u_y^2}}{\sin u}. \quad (1.3)$$

This equation is Darboux integrable and has Vessiot group  $\mathrm{SO}(3)$ . Since this group has no *real* 2-dimensional subgroups, we deduce that (1.3) cannot be related to the wave equation by a *real* Bäcklund transformation with 1-dimensional fibers. This conclusion is at odds with Theorem 1 of [14].

The paper is organized as follows. In Section 2 we review the basic properties of integrable extensions and reductions of differential systems. Detailed structure equations for the symmetry reduction of Pfaffian systems by free group actions are presented in Section 2.3. Theorem A is proved in Section 3. Darboux integrable differential systems are introduced in Section 4 and a simplified criteria for Darboux integrability is given. The first part of Theorem C is established in Section 5. In Section 5.2 we introduce the notion of *Darboux compatible* integrable extensions for Darboux integrable systems  $\mathcal{I}$  – these special extensions are required for Theorem C and are essential to our conclusions regarding (1.3). In Section 6 we generalize the group theoretic construction of Darboux integrable systems given in [3] to the case of reductions by non-diagonal group actions. Reductions by diagonal group actions are shown to give integrable extensions which are Darboux compatible in the sense introduced in Section 5.2.

In Section 7 the Vessiot algebra of a Darboux integrable system is defined. It is shown that the prolongation of a Darboux integrable system is Darboux integrable and that the Vessiot algebra is unchanged under prolongation. The Vessiot algebras for Darboux integrable systems constructed by symmetry reduction are given in Corollary 7.7. The second part of Theorem C and Theorem B are proved in Sections 8 and 9.

Examples are given in Section 10 and an application of the general theory developed in this article to the case of scalar Monge-Ampère equations in the plane is given in Section 11.

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## 2 Preliminaries

In this section we gather together a number of definitions and basic results on integrable extensions and reductions of exterior differential systems. Conventions and some results from [1] are used.

We assume that an EDS  $\mathcal{I}$ , defined on a manifold  $M$ , has constant rank in the sense that each one of its homogeneous components  $\mathcal{I}^p \subset \Omega^p(M)$  coincides with the sections  $\mathcal{S}(I^p)$  of a constant rank sub-bundle  $I^p \subset \Lambda^p(M)$ . If  $\mathcal{A}$  is a subset of  $\Omega^*(M)$ , we let  $\langle \mathcal{A} \rangle_{\text{alg}}$  and  $\langle \mathcal{A} \rangle_{\text{diff}}$  be the algebraic and differential ideals generated by  $\mathcal{A}$ . As usual, a differential system  $\mathcal{I}$  is called a Pfaffian system if there is a sub-bundle  $I \subset \Lambda^1(M)$  such that  $\mathcal{I} = \langle \mathcal{S}(I) \rangle_{\text{diff}}$ .

Let  $I \subset \Lambda^1(M)$  be a constant rank Pfaffian system. A **local first integral** of  $I$  is a smooth function  $f: U \rightarrow \mathbf{R}$ , defined on an open set  $U$ , and such that  $df \in I$ . For each point  $x \in M$  we define

$$I_x^\infty = \text{span}\{df_x \mid f \text{ is a local first integral, defined about } x\}. \quad (2.1)$$

We shall always assume that  $I^\infty = \cup_{x \in M} I_x^\infty$  is a constant rank bundle on  $M$ . It is easy to check that  $I^\infty$  is the (unique) maximal, completely integrable, Pfaffian subsystem of  $I$ .

The bundle  $I^\infty$  can be computed algorithmically from the derived flag of  $I$ . The derived system  $I' \subset I$  is defined pointwise by

$$I'_p = \text{span}\{\theta_p \mid \theta \in \mathcal{S}(I) \text{ such that } d\theta \equiv 0 \pmod{I}\}.$$

Letting  $I^{(0)} = I$  and assuming  $I^{(k)}$  is constant rank we define  $I^{(k+1)}$  inductively by  $I^{(k+1)} = (I^{(k)})'$  for  $k = 0, 1, \dots, N$ , where  $N$  is the smallest integer where  $I^{(N+1)} = I^{(N)}$ . Then  $I^{(N)}$  is completely integrable and  $I^\infty = I^{(N)}$ . More information about the derived flag of a Pfaffian system can be found in [5] and [24].

### 2.1 Integrable Extensions of Differential Systems

We recall the definition of an integrable extension [7]. First, if  $\mathcal{I}$  and  $\mathcal{J}$  are differential systems on the same manifold, we let  $\mathcal{I} + \mathcal{J} = \langle \mathcal{I} \cup \mathcal{J} \rangle_{\text{alg}}$ . Note that  $\mathcal{I} + \mathcal{J}$  is differentially closed. Let  $\mathbf{p}: N \rightarrow M$  be a surjective submersion and  $\mathcal{I}$  an EDS on  $M$ . An EDS  $\mathcal{E}$  on  $N$  is called an **integral extension** of  $\mathcal{I}$  if there exists a sub-bundle  $J \subset \Lambda^1(N)$  of rank  $\dim N - \dim M$ , such that

$$\text{ann}(J) \cap \ker(\mathbf{p}_*) = 0 \quad \text{and} \quad \mathcal{E} = \mathcal{S}(J) + \mathbf{p}^*(\mathcal{I}). \quad (2.2)$$

Here  $\text{ann}(J)$  is the sub-bundle of vectors in  $TM$  which annihilate the 1-forms in  $J$ . The first condition in (2.2) states that  $J$  is transverse to  $\mathbf{p}$ . A sub-bundle  $J$  satisfying the two properties (2.2) is called

an **admissible** sub-bundle for the extension  $\mathcal{E}$ . If  $\{\xi^u\}$  are 1-forms on  $N$  which define a local basis for  $\mathcal{S}(J)$ , then the second condition in (2.2) simply states that

$$d\xi^u \equiv 0 \pmod{\{\mathbf{p}^*(\mathcal{I}), \xi^u\}} \quad \text{or} \quad d\xi^u \equiv 0 \pmod{\{E^1, \mathbf{p}^*(I^2)\}}. \quad (2.3)$$

The first condition in (2.2) insures that if  $s : P \rightarrow N$  is an immersed integral manifold of  $\mathcal{E}$ , then  $\tilde{s} = \mathbf{p} \circ s : P \rightarrow M$  is an immersed integral manifold of  $\mathcal{I}$ . Conversely, if  $\tilde{s} : P \rightarrow M$  is an integral manifold of  $\mathcal{I}$  and  $Q = \mathbf{p}^{-1}(\tilde{s}(P))$ , then the second condition in (2.2) implies that  $\mathcal{E}|_Q$  is a Frobenius system.

A few remarks concerning this definition are in order.

**IE [i]** If  $\mathcal{E}^1 = \mathcal{S}(E^1)$  and  $\mathcal{I}^1 = \mathcal{S}(I^1)$ , then equations (2.2) implies that

$$E^1 = J \oplus \mathbf{p}^*(I^1) \quad \text{and} \quad \Lambda^1(N) = J \oplus \mathbf{p}^*(\Lambda^1(M)). \quad (2.4)$$

**IE [ii]** If  $\mathbf{p} : \mathcal{E} \rightarrow \mathcal{I}$  is an integrable extension, then *any* complementary sub-bundle  $J$  to  $\mathbf{p}^*(I^1)$  in  $E^1$  satisfies (2.2) and is therefore admissible.

**IE [iii]** If  $\mathcal{E}$  is an integral extension of  $\mathcal{I}$  with respect to the submersion  $\mathbf{p}$ , then  $\mathcal{E}/\mathbf{p} = \mathcal{I}$ .

**IE [iv]** If  $\mathcal{I}$  is a Pfaffian system, then  $\mathcal{E}$  is a Pfaffian system with  $E = J \oplus \mathbf{p}^*(I)$  and

$$\text{rank}(E) = \text{rank}(I) + v \quad \text{and} \quad \text{rank}(E') = \text{rank}(I') + v, \quad (2.5)$$

where  $v$  is the dimension of the fibre of  $\mathbf{p}$ , that is,  $v = \text{rank}(\ker(\mathbf{p}_*)) = \text{rank } J$ .

**IE [v]** If  $\mathcal{I}$  is a completely integrable Pfaffian system, then equation (2.5) implies that  $\mathcal{E}$  is completely integrable. Moreover, if  $s : P \rightarrow N$  is the maximal integral manifold through  $x_0 \in N$  for  $\mathcal{E}$  and  $\tilde{s} : \tilde{P} \rightarrow M$  is the maximal integral manifold through  $\mathbf{p}(x_0) \in M$  for  $\mathcal{I}$ , then  $\mathbf{p} \circ s : P \rightarrow \tilde{s}(\tilde{P})$  is a local diffeomorphism.

**IE [vi]** Suppose the Pfaffian system  $\mathcal{I}$  satisfies  $(I^1)^\infty = 0$ . Then, without loss of generality, it can be assumed that the integral extension  $\mathcal{E}$  also satisfies  $(E^1)^\infty = 0$ . Indeed, let  $s : P \rightarrow N$  be a maximal integral manifold of  $(\mathcal{E}^1)^\infty$ , and let  $\pi : N \rightarrow M$  be  $\pi = \mathbf{p} \circ s$ . Then  $\mathcal{K} = s^*(\mathcal{E})$  is an integrable extension of  $\mathcal{I}$  on  $P$  satisfying  $(K^1)^\infty = 0$ .

**IE [vii]** Let  $\phi : (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension with  $J \subset T^*N$  an admissible bundle and let  $\pi_N : (\mathcal{E}^{[1]}, N^{[1]}) \rightarrow (\mathcal{E}, N)$  and  $\pi_M : (\mathcal{I}^{[1]}, M^{[1]}) \rightarrow (\mathcal{I}, M)$  be the prolongations of  $\mathcal{E}$  and  $\mathcal{I}$ . Then  $\phi$  lifts to a unique map  $\phi^{[1]} : (\mathcal{E}^{[1]}, N^{[1]}) \rightarrow (\mathcal{I}^{[1]}, M^{[1]})$  which covers  $\phi$  and  $\mathcal{E}^{[1]}$  is an integrable extension of  $\mathcal{I}^{[1]}$  with admissible bundle  $\pi_N^*(J)$ .

## 2.2 Reduction of Differential Systems

Recall from the introduction that if  $\mathbf{p} : M \rightarrow N$  is a smooth surjective submersion and  $\mathcal{I}$  an EDS on  $N$ , then the **reduction of**  $\mathcal{I}$  with respect to  $\mathbf{p}$  which is denoted by  $\mathcal{I}/\mathbf{p}$ , is the EDS on  $M$  defined by

$$\mathcal{I}/\mathbf{p} = \{ \theta \in \Omega^*(M) \mid \mathbf{p}^*\theta \in \mathcal{I} \}. \quad (2.6)$$

Note that  $\mathcal{I}/\mathbf{p}$  is not necessarily constant rank without additional hypothesis. If the fibers of  $\mathbf{p}$  are connected, then the reduction  $\mathcal{I}/\mathbf{p}$  can be computed using Corollary 2.3 of [5] which states that  $\theta \in \mathcal{I}$  satisfies  $\theta = \mathbf{p}^*\bar{\theta}$  for some  $\bar{\theta} \in \mathcal{I}/\mathbf{p}$  if and only if,

$$X \lrcorner \theta = 0, \quad X \lrcorner d\theta = 0 \quad \text{for all } X \in \ker(\mathbf{p}_*). \quad (2.7)$$

We now specialize to the case of reduction by a Lie symmetry group. Let  $G$  be a finite dimensional Lie group acting on  $M$  with action  $\mu : G \times M \rightarrow M$ . Let  $\Gamma_G$  be the Lie algebra of infinitesimal generators for the action  $\mu$  and let  $\Gamma_G \subset TM$  be the integrable distribution generated by the point-wise span of  $\Gamma_G$ . We will assume that all actions are regular in the sense that the orbit space  $M/G$  has a smooth manifold structure such that the canonical projection  $\mathbf{q}_G : M \rightarrow M/G$  is a smooth submersion and  $\Gamma_G = \ker(\mathbf{q}_{G*})$ .

The group  $G$  acting on  $M$  is a **symmetry group of  $\mathcal{I}$**  if, for each  $g \in G$  and  $\theta \in \mathcal{I}$ ,  $\mu_g^*(\theta) \in \mathcal{I}$ . Under these circumstances we define the **symmetry reduction of  $\mathcal{I}$  by  $G$**  to be

$$\mathcal{I}/G = \mathcal{I}/\mathbf{q}_G = \{ \bar{\theta} \in \Omega^*(M/G) \mid \mathbf{q}_G^*(\bar{\theta}) \in \mathcal{I} \}. \quad (2.8)$$

In other words  $\mathcal{I}/G$  is the reduction given by equation (2.6) with respect to the submersion  $\mathbf{q}_G : M \rightarrow M/G$ . However, by utilizing the  $G$ -invariance of  $\mathcal{I}$ , the computation of a local basis of sections for  $\mathcal{I}/G$  can now be done algebraically [1]. This fact is not necessarily true for the reduction in (2.6) for generic  $\mathbf{p}$ . In analogy with (2.7), a form  $\theta \in \Omega^p(M)$  satisfies  $\theta = \mathbf{q}_G^*(\bar{\theta})$  for some  $\bar{\theta} \in \Omega^p(M/G)$  if and only if  $\theta$  is  **$G$ -basic**, that is,  $G$  semi-basic and  $G$  invariant so that

$$X \lrcorner \theta = 0 \quad \text{and} \quad \mu_g^*(\theta) = \theta \quad \text{for all } X \in \Gamma_G \text{ and } g \in G. \quad (2.9)$$

A symmetry group  $G$  of an EDS  $\mathcal{I}$  is said to be **transverse to  $\mathcal{I}$**  if

$$\text{ann}(I^1) \cap \Gamma_G = 0, \quad (2.10)$$

The actions considered in this paper all satisfy the transversality condition (2.10). We shall summarize a few relevant facts from [1] about symmetry reduction by transverse actions.

First, for any sub-bundle  $A^p \subset \Lambda^p(M)$ , we define  $A_{G,\text{sb}}^p$  to be the  $G$  semi-basic forms, that is,

$$A_{G,\text{sb}}^p = \{ \alpha \in A^p \mid X \lrcorner \alpha = 0 \text{ for all } X \in \Gamma_G \}. \quad (2.11)$$

It is clear that if  $A^p$  is  $G$ -invariant, then  $A_{G,\text{sb}}^p$  is  $G$ -invariant. If, in addition,  $A_{G,\text{sb}}^p$  is a constant-rank bundle then there is a bundle  $\bar{A}^p \subset \Lambda^p(M/G)$ , of the same rank as  $A_{G,\text{sb}}^p$ , satisfying

$$\mathbf{q}_G^*(\bar{A}^p) = A_{G,\text{sb}}^p. \quad (2.12)$$

Furthermore, about each point  $x \in M$ , there is an  $G$ -invariant open set  $U$  and a local basis  $\{\alpha^i\}$  for  $\bar{A}^p|_{\mathbf{q}_G(U)}$  such that  $\{\mathbf{q}_G^*(\alpha^i)\}$  is a local basis for  $A_{G,\text{sb}}^p$ .

In the particular case of interest, namely when  $G$  is symmetry group of a differential system  $\mathcal{I}$  and  $G$  is transverse to  $\mathcal{I}$ , then these observations apply to the homogeneous components  $I^p$ . However, as shown in [1], transversality now automatically insures that spaces  $I_{G,\mathbf{sb}}^p$  are all constant rank with, in particular,

$$\text{rank}(I^1/G) = \text{rank}(I_{G,\mathbf{sb}}^1) = \text{rank}(I^1) - \text{rank}(\Gamma_G). \quad (2.13)$$

Consequently, the quotient  $\mathcal{I}/G$  is a constant rank EDS.

Algebraic generators for  $\mathcal{I}$  adapted to the  $G$ -action are obtained as follows. Choose bundles  $J$  and a  $G$ -invariant,  $G$  semi-basic bundle  $W_{G,\mathbf{sb}}$  such that

$$I_{G,\mathbf{sb}}^1 \oplus J = I^1 \quad \text{and} \quad I_{G,\mathbf{sb}}^1 \oplus W_{G,\mathbf{sb}} = \Lambda_{G,\mathbf{sb}}^1(M).$$

It is a simple matter to show that  $\Lambda_{G,\mathbf{sb}}^1(M) \oplus J = \Lambda^1(M)$  in which case

$$I_{G,\mathbf{sb}}^1 \oplus J \oplus W_{G,\mathbf{sb}} = \Lambda^1(M).$$

Then, about each point  $x \in M$  there is an open set  $U$  in  $M$  with

$$\mathcal{I}|_U = \langle \theta^i, \theta_G^a, \tau_G^\alpha \rangle_{\text{alg}} \quad \text{and} \quad (\mathcal{I}/G)|_{\mathbf{q}_G(U)} = \langle \bar{\theta}^a, \bar{\tau}^\alpha \rangle_{\text{alg}}, \quad \text{where} \quad (2.14)$$

$$J|_U = \text{span}\{\theta^i\}, \quad I_{G,\mathbf{sb}}^1|_U = \text{span}\{\theta_G^a = \mathbf{q}_G^* \bar{\theta}^a\} \quad \text{and} \quad \tau_G^\alpha = \mathbf{q}_G^* \bar{\tau}^\alpha \in \Omega^*(W_{G,\mathbf{sb}}|_U).$$

By definition,  $\mathcal{I}$  is differentially closed and hence

$$d\theta^i \equiv 0 \mod \langle \theta^i, \theta_G^a, \tau_G^\alpha \rangle_{\text{alg}}. \quad (2.15)$$

We shall give a refinement of this structure equations for the case of free group actions in section 3.3.

We shall need the following.

**Theorem 2.1.** *Let  $I$  be a Pfaffian system on a manifold  $M$ . Suppose  $G$  acts regularly on  $M$  with quotient map  $\mathbf{q}_G: M \rightarrow M/G$ ,  $G$  is a symmetry group of  $I$ , and  $G$  acts transversally to  $I$ . If  $(I^\infty)_{G,\mathbf{sb}}$  and  $(I/G)^\infty$  are constant rank vector bundles, then*

$$(I^\infty)_{G,\mathbf{sb}} = \mathbf{q}_G^*((I/G)^\infty) \quad \text{and} \quad (I/G)^\infty = I^\infty/G. \quad (2.16)$$

*Proof.* We prove the second equation in (2.16). The first second then follows from (2.12). We first remark that the transversality hypothesis implies that  $I/G$  has constant rank. Since the pullback of an integrable Pfaffian system is integrable and  $\mathbf{q}_G^*$  is injective, it follows that  $\mathbf{q}_G^*((I/G)^\infty)$  is a constant rank integrable sub-bundle of  $I$  and therefore  $\mathbf{q}_G^*((I/G)^\infty) \subset I^\infty$ . By definition (2.8), this implies that

$$(I/G)^\infty \subset I^\infty/G.$$

To prove the reverse inclusion we deduce, from the assumption that  $(I^\infty)_{G,\mathbf{sb}}$  is of constant rank, that is,  $I^\infty/G$  is a constant rank sub-bundle of  $I/G$ . But  $I^\infty/G$  is integrable [15] and hence  $I^\infty/G \subset (I/G)^\infty$  and the lemma is established.  $\blacksquare$

For additional results on symmetry reduction of derived systems, see [4].



### 2.3 Symmetry Reduction of Pfaffian Systems by Free Group Actions

It will be important to have a refinement of the foregoing general results on reduction of differential systems for the special case where the action of  $G$ , in addition to being regular and transverse, is also free. In other words, we now consider a Pfaffian system  $I$  defined on a left principle  $G$  bundle  $M$ . Recall that a local trivialization for  $\mathbf{q}_G : M \rightarrow M/G$  consists of an  $G$  invariant open set  $U \subset M$  and a diffeomorphism  $\Phi : U \rightarrow \mathbf{q}_G(U) \times G$  where  $\Phi(x) = (\mathbf{q}_G(x), \phi(x))$  and  $\phi : U \rightarrow G$  is  $G$ -equivariant, that is,  $\phi(\mu(g, x)) = g\phi(x)$ .

For the  $r$ -dimensional Lie group  $G$  we make the following choices. Let  $Z_i$ ,  $1 \leq i \leq r$  be a basis of right invariant vector fields, and let  $\tau^i$  be the dual right invariant one-forms to  $Z_i$  on  $G$ . Let  $[Z_i, Z_j] = C_{ij}^k Z_k$ . Define the matrix-valued function  $\lambda : G \rightarrow GL(r, \mathbf{R})$  by

$$\text{Ad}^*(g)(\tau^i) = \lambda(g)_j^i \tau^j, \quad (2.17)$$

where  $\text{Ad}^*$  is the co-adjoint representation of  $G$ . Equation (2.17) implies that

$$L_g^* \tau^i = \lambda(g^{-1})_j^i \tau^j, \quad (2.18)$$

where  $L_g$  is left multiplication on  $G$  by  $g$ . Since  $\text{Ad}^*(g g') = \text{Ad}^*(g) \text{Ad}^*(g')$  we also deduce that

$$\lambda(g g')_j^i = \lambda(g)_j^k \lambda(g')_k^i. \quad (2.19)$$

The differential of  $\text{Ad}$  is  $\text{ad}$  and so the exterior derivative of (2.17) gives

$$d\lambda_j^i = \lambda_k^i C_{lj}^k \tau^l. \quad (2.20)$$

In this appendix we provide the missing details for Section 2.2. We begin with the proof of equation (2.20). With  $\{Z_i\}$  the basis of right invariant vector-fields we first compute  $\lambda_* Z_i(e)$  by

$$\begin{aligned} \frac{d}{dt} \lambda(\exp(tZ_i))(\omega^j(e))|_{t=0} &= \frac{d}{dt} (L_{\exp(-tZ_i)}^* \circ R_{\exp(tZ_i)}^*)(\omega^j(e))|_{t=0} \\ &= (-\mathcal{L}_{Z_i} \omega^j)(e) = C_{ik}^j \omega^k(e) \end{aligned} \quad (2.21)$$

Therefore

$$d\lambda_j^i(e) = C_{kj}^i \omega^k.$$

Then we may use property 2.42 (2.19) to compute  $\lambda_*(Z_i(a))$ . Since  $Z_i$  is a right invariant vector-field we have

$$\frac{d}{dt} \lambda(\exp(tZ_i a))_j^i|_{t=0} = \lambda_k^i(a) \frac{d}{dt} \lambda_j^k(\exp(tZ_i)|_{t=0}) = \lambda_k^i(a) C_{ij}^k \quad (2.22)$$

which establishes (2.20).

**Theorem 2.2.** *Let  $\mathcal{I}$  be a Pfaffian system on a manifold  $M$ . Let  $G$  be a symmetry group of  $\mathcal{I}$  which acts freely and regularly on  $M$  and transversely to  $\mathcal{I}$ . Let  $\{X_1, X_2, \dots, X_r\}$  be a basis of infinitesimal generators for the action of  $G$  on  $M$  corresponding to the choice of right invariant vector-fields  $Z_i$*

above, with structure equations  $[X_i, X_j] = C_{ij}^k X_k$ . Then about each point  $x \in M$  there exists a  $G$ -invariant open set  $U$  and a co-frame  $\{\theta^i, \eta^a, \sigma^\alpha\}$  on  $U$  such that:

[i]  $\mathcal{I} = \langle \theta^i, \eta^a \rangle_{\text{diff}}$ ;

[ii] the forms  $\eta^a, \sigma^\alpha$  are  $G$  basic;

[iii] the forms  $\theta^i$  satisfy  $\theta^i(X_j) = \delta_j^i$ ,

[iv]  $\mu_g^* \theta^i = \lambda(g^{-1})_j^i \theta^j$ , where  $\lambda_j^i(g)$  is the matrix defined in equation (2.17); and

[v] where the structure equations are

$$\begin{aligned} d\sigma^\alpha &= 0, \quad d\eta^a \equiv 0 \pmod{\{\sigma^\alpha, \eta^a\}}, \\ d\theta^i &= A_{\alpha\beta}^i \sigma^\alpha \wedge \sigma^\beta + B_{a\beta}^i \eta^a \wedge \sigma^\beta - \frac{1}{2} C_{jk}^i \theta^j \wedge \theta^k. \end{aligned} \tag{2.23}$$

*Proof.* Let  $n = \dim(M)$ ,  $r = \dim(G)$ , and  $p = \text{rank } I^1$ . Again, because the action of  $G$  is free and regular, we may choose a local trivialization with open set  $U$  and map  $(\mathbf{q}_G, \phi) : U \rightarrow \bar{U} \times G$  where  $\bar{U}$  is a coordinate chart on  $\bar{M}$ . Let  $J^1, J^2, \dots, J^{n-r}$  be a set of coordinates on  $\bar{U}$ , whose pullback to  $U$  which again we denote by  $J^1, \dots, J^{n-r}$  are  $G$ -invariant. Since the rank of  $T^*M_{G,\mathbf{sb}}$  is  $n - r$ , the differentials of these invariant functions give a basis for  $\Omega^1(U)_{G,\mathbf{sb}}$ .

On account of (2.14), we may assume that on the  $G$ -invariant open set  $U$ ,  $\{\eta^1, \eta^2, \dots, \eta^{p-r}\}$  is a basis of  $G$ -invariant sections for  $I_{G,\mathbf{sb}}^1$ , restricted to  $U$ . Since  $I_{G,\mathbf{sb}}^1 \subset T^*M_{G,\mathbf{sb}}$ , we can choose a set of differentials  $\sigma^\alpha = dJ^{k_\alpha}$  complementary to the 1-forms  $\eta^a$ . This gives  $\Omega^1(U)_{G,\mathbf{sb}} = \text{span}\{\eta^a, \sigma^\alpha\}$ . The forms  $\sigma^\alpha$  are closed, by construction, while the second set of structure equations follow from the fact that the forms  $\eta^a$  are  $G$ -basic.

Let  $\omega^i = \phi^*(\tau^i)$ , where  $\tau^i$  are the right-invariant forms on  $G$  defined above and dual to  $Z_i$ . Note that  $\omega^i(X_j) = \delta_j^i$ , and

$$d\omega^i = -\frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k.$$

The forms  $\{\omega^i, \eta^a, \sigma^\alpha\}$  are a basis of sections of  $T^*U$ .

Now choose 1-forms  $\{\tilde{\theta}^1, \dots, \tilde{\theta}^p\}$  so that  $\{\tilde{\theta}^i, \eta^a\}$  is a basis of sections for  $I^1$ , restricted to  $U$ . We can write

$$\tilde{\theta}^i = \tilde{P}_j^i \omega^j + \tilde{S}_a^i \eta^a + \tilde{T}_\alpha^i \sigma^\alpha. \tag{2.24}$$

Transversality implies that the matrix  $\tilde{P}_j^i$  is invertible and therefore there are smooth functions  $S_a^i, T_\alpha^i$  on  $U$  such that, for each  $i = 1 \dots r$ ,

$$\omega^i + S_a^i \eta^a + T_\alpha^i \sigma^\alpha \in \mathcal{I}. \tag{2.25}$$

Since  $\eta^a \in \mathcal{I}$ , this implies that

$$\theta^i = \omega^i + T_\alpha^i \sigma^\alpha \in \mathcal{I}. \tag{2.26}$$

At this point the 1-forms  $\{\theta^i, \eta^a, \sigma^\alpha\}$  satisfy parts [i], [ii], [iii] of the theorem.

To show [iv] first note that by the equivariance of  $\phi$  and equation (2.18),

$$\mu_g^*(\omega^i) = \mu_g^*\phi^*(\tau^i) = \phi^*L_g^*\lambda(g^{-1})^i_j\omega^j. \quad (2.27)$$

Now using the  $G$ -invariance of  $\sigma^\alpha$  and equation (2.27) we have

$$\mu_g^*\theta^i(gp) = \lambda(g^{-1})^i_j\omega^j(p) + T_\alpha^i(gp)\sigma^\alpha(p). \quad (2.28)$$

Since  $\mathcal{I}$  is invariant  $\mu_g^*\theta^i \in \mathcal{S}(I^1|_U)$ , and equation (2.28) leads to

$$\mu_g^*\theta^i = \lambda(g^{-1})^i_j\theta^j \quad \text{and} \quad T_\alpha^i(gp) = \lambda(g^{-1})^i_jT_\alpha^j(p).$$

This proves property [iv].

Finally we complete the proof of part [v] of the theorem. By direct calculation using equation (2.26), we have

$$\begin{aligned} d\theta^i &= d\omega^i + dT_\alpha^i\sigma^\alpha = -\frac{1}{2}C_{jk}^i\omega^j \wedge \omega^k + dT_\alpha^i \wedge \sigma^\alpha \\ &= -\frac{1}{2}C_{jk}^i\theta^j \wedge \theta^k + C_{jk}^i\lambda_\beta^k\theta^j \wedge \sigma^\beta - \frac{1}{2}C_{jk}^iT_\alpha^jT_\beta^k\sigma^\alpha \wedge \sigma^\beta + dT_\alpha^i \wedge \sigma^\alpha. \end{aligned}$$

There are no 2-forms of the kind  $\theta^i \wedge \eta^a$  in this formula for  $d\theta^i$ . Moreover, since the vector fields  $X_\ell$  are infinitesimal symmetries for  $\mathcal{I}$ , it follows that  $X_\ell \lrcorner d\theta^i \in \mathcal{I}$  and therefore the terms of the kind  $\theta^i \wedge \sigma^\alpha$  (appearing in  $C_{jk}^iT_\beta^k\theta^j \wedge \sigma^\beta$  and  $dT_\alpha^i \wedge \sigma^\alpha$ ) must cancel out. This leads to the structure equations (2.23) for the 1-forms  $\theta^i$ . ■

### 3 Reduction of Differential Systems

In this section we shall prove Theorem A.

#### 3.1 Reductions of EDS and commutative diagrams

We begin with a theorem which shows that reduction of EDS behaves well with respect to commutative diagrams.

**Theorem 3.1.** *Let*

$$\begin{array}{ccc} P & \xrightarrow{\mathbf{p}_1} & N \\ & \searrow \mathbf{p}_3 & \downarrow \mathbf{p}_2 \\ & & M \end{array} \quad (3.1)$$

be a commutative diagram of manifolds and suppose that  $\mathbf{p}_1$  and  $\mathbf{p}_3$  are surjective submersions. Then  $\mathbf{p}_2$  is a surjective submersion and

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathbf{p}_1} & \mathcal{I}/\mathbf{p}_1 \\ & \searrow \mathbf{p}_3 & \downarrow \mathbf{p}_2 \\ & & \mathcal{I}/\mathbf{p}_3 \end{array} \quad (3.2)$$

is a commutative diagram of EDS.

*Proof.* The fact that  $\mathbf{p}_2$  is a surjective submersion is easily checked. The definitions of  $(\mathcal{I}/\mathbf{p}_1)/\mathbf{p}_2$ ,  $\mathcal{I}/\mathbf{p}_1$  and  $\mathcal{I}/\mathbf{p}_3$  imply, in turn, that

$$\begin{aligned} (\mathcal{I}/\mathbf{p}_1)/\mathbf{p}_2 &= \{ \theta \in \Omega^*(M) \mid \mathbf{p}_2^* \theta \in \mathcal{I}/\mathbf{p}_1 \} = \{ \theta \in \Omega^*(M) \mid \mathbf{p}_1^*(\mathbf{p}_2^* \theta) \in \mathcal{I} \} \\ &= \{ \theta \in \Omega^*(M) \mid \mathbf{p}_3^* \theta \in \mathcal{I} \} = \mathcal{I}/\mathbf{p}_3 \end{aligned}$$

which establishes the commutativity of the EDS reduction diagram (3.2).  $\blacksquare$

For future reference, we remark that the commutative diagram (3.1) also implies that

$$\ker \mathbf{p}_{2*} = \mathbf{p}_{1*}(\ker \mathbf{p}_{3*}). \quad (3.3)$$

**Corollary 3.2.** *Let  $G$  be a symmetry group of an EDS  $\mathcal{I}$  on a manifold  $M$  which acts regularly on  $M$ . Let  $H \subset G$  be a subgroup of  $G$  which also acts regularly on  $M$ . Then the orbit mapping*

$$\mathbf{p} : M/H \rightarrow M/G \quad \text{defined by} \quad \mathbf{p}(Hx) = Gx \quad (3.4)$$

*is a surjective submersion which gives rise to the following commutative diagram of EDS*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathbf{q}_H} & \mathcal{I}/H \\ & \searrow \mathbf{q}_G & \downarrow \mathbf{p} \\ & & \mathcal{I}/G, \end{array} \quad (3.5)$$

*that is,  $(\mathcal{I}/H)/\mathbf{p} = \mathcal{I}/G$ .*

*Proof.* The given hypothesis leads immediately to the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathbf{q}_H} & M/H \\ & \searrow \mathbf{q}_G & \downarrow \mathbf{p} \\ & & M/G, \end{array}$$

where  $\mathbf{q}_G$  and  $\mathbf{q}_H$  are surjective submersions. Theorem 3.1 then implies that  $\mathbf{p}$  is a surjective submersion and that the diagram (3.5) commutes.  $\blacksquare$

**Proof of part [i] of Theorem A:** Apply Corollary 3.2 to each of the two cases  $H \subset G_1$  and  $H \subset G_2$  and this immediately produces the commutative diagram (1.2).

**Remark 3.3.** Let  $G$  act freely and regularly on  $M$ . If  $K \subset G$  is a closed subgroup, then  $K$  acts freely and regularly on  $M$ . If, in addition,  $K$  is a normal subgroup of  $G$ , then the natural action of  $G/K$  on  $M/K$  is both free and regular. If  $G$  is a symmetry group of  $\mathcal{I}$ , then  $G/K$  is a symmetry group of  $\mathcal{I}/K$ . By Corollary 3.2,

$$\mathcal{I}/G = (\mathcal{I}/K)/(G/K),$$

where the projection map  $\mathbf{p}$  in (3.5) is now the quotient map for the action of  $G/K$  on  $M/K$ . Finally, if  $G$  acts transversally to  $\mathcal{I}$  then  $G/K$  acts transversally to  $\mathcal{I}/K$ .

### 3.2 Subgroup Reductions and Integrable Extensions

The proof of part [ii] of Theorem A follows directly from the following general result.

**Theorem 3.4.** *Let  $G$  be a symmetry group of an EDS  $\mathcal{I}$  on a manifold  $M$ . Assume  $G$  acts regularly on  $M$  and transversally to  $\mathcal{I}$ , and let  $H \subset G$  be a Lie subgroup of  $G$  which also acts regularly on  $M$ . Then the orbit mapping  $\mathbf{p}$  in (3.5) defines  $\mathcal{I}/H$  as an integrable extension of  $\mathcal{I}/G$ .*

*Proof.* The transversality assumptions on  $G$  and  $H$  insure that the quotient bundles  $I^1/G$  and  $I^1/H$  are constant rank and consequently we may construct a constant rank bundle  $\bar{K} \subset \Lambda^1(M/H)$  such that

$$I^1/H = \bar{K} \oplus \mathbf{p}^*(I^1/G). \quad (3.6)$$

We shall show that

$$\text{ann}(\bar{K}) \cap \ker(\mathbf{p}^*) = 0 \quad \text{and} \quad \mathcal{I}/H = \mathcal{S}(\bar{K}) + \mathbf{p}^*(\mathcal{I}/G). \quad (3.7)$$

so that  $\bar{K}$  is an admissible bundle which defines  $\mathcal{I}/H$  as an integrable extension of  $\mathcal{I}/G$ . To begin, we first note that: (i) the application of (3.3) to the diagram (3.5) yields

$$\ker(\mathbf{p}_*) = \mathbf{q}_{H*}(\Gamma_G); \quad (3.8)$$

so that the first condition in (3.7) is equivalent to

$$\mathbf{q}_{H*}(\text{ann}(K_{H,\mathbf{sb}}) \cap \Gamma_G) = 0 \quad \text{where} \quad K_{H,\mathbf{sb}} = \mathbf{q}_H^*(\bar{K}); \quad (3.9)$$

(ii) the commutativity of (3.5), together with (2.12), implies that

$$I_{H,\mathbf{sb}}^1 = I_{G,\mathbf{sb}}^1 \oplus K_{H,\mathbf{sb}} \quad (3.10)$$

and; (iii) transversality (see (2.13)) gives

$$\text{rank}(K_{H,\mathbf{sb}}) = \text{rank } I_{H,\mathbf{sb}}^1 - \text{rank } I_{G,\mathbf{sb}}^1 = \text{rank } \Gamma_G - \text{rank } \Gamma_H.$$

This, in turn, implies that

$$\text{rank}(\Lambda_{H,\text{sb}}^1(M)) = \Lambda^1(M) - \text{rank} \Gamma_H = \text{rank}(\Lambda_{G,\text{sb}}^1(M)) + \text{rank}(K_{H,\text{sb}}) \quad (3.11)$$

and hence  $\Lambda_{H,\text{sb}}^1(M) = \Lambda_{G,\text{sb}}^1(M) \oplus K_{H,\text{sb}}$ . Now choose complementary bundles  $W_{G,\text{sb}}$  and  $L$  such that

$$I_{G,\text{sb}}^1 \oplus W_{G,\text{sb}} = \Lambda_{G,\text{sb}}^1(M) \quad \text{and} \quad I_{H,\text{sb}}^1 \oplus L = I^1. \quad (3.12)$$

in which case

$$\Lambda_{H,\text{sb}}^1(M) = \Lambda_{G,\text{sb}}^1(M) \oplus K_{H,\text{sb}} = I_{G,\text{sb}}^1 \oplus W_{G,\text{sb}} \oplus K_{H,\text{sb}}. \quad (3.13)$$

This equation shows that

$$\begin{aligned} \Gamma_G \cap \text{ann}(K_{H,\text{sb}}) &= \text{ann}(I_{G,\text{sb}}^1 \oplus W_{G,\text{sb}}) \cap \text{ann}(K_{H,\text{sb}}) = \text{ann}(I_{G,\text{sb}}^1 \oplus W_{G,\text{sb}} \oplus K_{H,\text{sb}}) \\ &= \text{ann}(\Lambda_{H,\text{sb}}^1(M)) = \Gamma_H \end{aligned}$$

which immediately implies (3.9).

To check the second equation in (3.7), we shall refine the local co-frames (2.14) using (3.10) - (3.13). Since  $I_{H,\text{sb}}^1 \oplus L = I^1 = I_{G,\text{sb}}^1 \oplus J$ , we can deduce from (3.10) that

$$J = K_{H,\text{sb}} \oplus L \quad \text{and hence} \quad I_{G,\text{sb}}^1 \oplus K_{H,\text{sb}} \oplus L \oplus W_{G,\text{sb}} = \Lambda^1(M).$$

Accordingly, about each point of  $M$ , there is an open set  $U$  and forms  $\theta^u, \theta_H^r, \theta_G^a, \tau_G^\alpha$  with

$$L|_U = \text{span}\{\theta^u\}, \quad K_{H,\text{sb}}|_U = \text{span}\{\theta_H^r = \mathbf{q}_H^* \tilde{\theta}^r\}, \quad I_{G,\text{sb}}^1|_U = \text{span}\{\theta_G^a = \mathbf{q}_G^* \tilde{\theta}^a\},$$

$$\mathcal{I}|_U = \langle \theta^u, \theta_H^r, \theta_G^a, \tau_G^\alpha \rangle_{\text{alg}}, \quad \bar{K}|_{\mathbf{q}_H(U)} = \text{span}\{\tilde{\theta}^r\} \quad \text{and} \quad (\mathcal{I}/G)|_{\mathbf{q}_G(U)} = \langle \tilde{\theta}^a, \tilde{\tau}^\alpha \rangle_{\text{alg}},$$

where  $\tau_G^\alpha = \mathbf{q}_G(\tilde{\tau}^\alpha) \in \Lambda^*(W_{G,\text{sb}})$ . Put  $\tilde{\theta}^a = \mathbf{p}^*(\tilde{\theta}^a)$  and  $\tilde{\tau}^\alpha = \mathbf{p}^*(\tilde{\tau}^\alpha)$ . Then, in terms of these forms, we have

$$\langle \mathbf{p}^*(\mathcal{I}/G) \rangle_{\text{alg}}|_{\mathbf{q}_H(U)} = \langle \tilde{\theta}^a, \tilde{\tau}^\alpha \rangle_{\text{alg}} \quad \text{and} \quad (\mathcal{I}/H)|_{\mathbf{q}_H(U)} = \langle \tilde{\theta}^r, \tilde{\theta}^a, \tilde{\tau}^\alpha \rangle_{\text{alg}} \quad (3.14)$$

and the second equation in (3.7) is proved. ■

**Corollary 3.5.** *Let  $G$  be a symmetry group of an EDS  $\mathcal{I}$  on a manifold  $M$ . Assume  $G$  acts regularly on  $M$  and transversally to  $\mathcal{I}$ . Then  $\mathcal{I}$  is an integral extension of the reduced differential system  $\mathcal{I}/G$ .*

**Corollary 3.6.** *Under the hypothesis of Theorem 3.4, the diagram (3.5) is a commutative diagram of integrable extensions.*

**Proof of part [ii] of Theorem A:** The transversality hypothesis in part [ii] of Theorem A allows to apply Corollaries 3.5 and 3.6 to the two cases  $H \subset G_1$  and  $H \subset G_2$ .

We remark that  $\mathcal{I}$  is a rather special type of integrable extension of  $\mathcal{I}/G$  and perhaps deserving of the designation *integrable extension of Lie type*. Indeed, it is shown in [1], in the special case where  $\mathcal{I}$  is Pfaffian and the action of  $G$  is free, that the integral manifolds for  $\mathcal{I}$  can be constructed from the integral manifolds of  $\mathcal{I}/G$  by the integration of ODE's of Lie type.

## 4 Darboux Integrable Differential Systems

In this section we review the definition of Darboux integrable differential systems and we state a new result which simplifies the conditions of this definition. We begin with the definition of a decomposable differential system and the notion of a Darboux pair as found in [3].

**Definition 4.1.** *An exterior differential system  $\mathcal{I}$  on  $M$  is **decomposable of type**  $[p, \rho]$ , where  $p, \rho \geq 2$ , if about each point  $x \in M$  there is a local co-frame*

$$\tilde{\theta}^1, \dots, \tilde{\theta}^r, \hat{\sigma}^1, \dots, \hat{\sigma}^p, \check{\sigma}^1, \dots, \check{\sigma}^\rho, \quad (4.1)$$

*such that  $\mathcal{I}$  is algebraically generated by 1-forms and 2-forms*

$$\mathcal{I} = \langle \tilde{\theta}^1, \dots, \tilde{\theta}^r, \hat{\Omega}^1, \dots, \hat{\Omega}^s, \check{\Omega}^1, \dots, \check{\Omega}^\tau \rangle_{\text{alg}}, \quad (4.2)$$

*where  $s, \tau \geq 1$ ,  $\hat{\Omega}^a \in \Omega^2(\hat{\sigma}^1, \dots, \hat{\sigma}^p)$ , and  $\check{\Omega}^\alpha \in \Omega^2(\check{\sigma}^1, \dots, \check{\sigma}^\rho)$ . The differential systems algebraically generated by*

$$\hat{\mathcal{V}} = \langle \tilde{\theta}^e, \hat{\sigma}^a, \check{\Omega}^\alpha \rangle_{\text{alg}} \quad \text{and} \quad \check{\mathcal{V}} = \langle \tilde{\theta}^e, \check{\sigma}^\alpha, \hat{\Omega}^a \rangle_{\text{alg}} \quad (4.3)$$

*are called the associated **singular differential systems** for  $\mathcal{I}$  with respect to the decomposition (4.2), while the bundles  $\hat{V}, \check{V} \subset T^*M$  defined by*

$$\hat{V} = \{ \tilde{\theta}^e, \hat{\sigma}^a \} \quad \text{and} \quad \check{V} = \{ \tilde{\theta}^e, \check{\sigma}^\alpha \} \quad (4.4)$$

*are called the **associated singular Pfaffian systems**.*

Equation (4.2) implies that the 1-forms  $\tilde{\theta}^e$  satisfy structure equations of the form

$$d\tilde{\theta}^e \equiv A_{ab}^e \hat{\sigma}^a \wedge \hat{\sigma}^b + B_{\alpha\beta}^e \hat{\sigma}^\alpha \wedge \hat{\sigma}^\beta \quad \text{mod } \{ \tilde{\theta}^e \}. \quad (4.5)$$

In particular, any class  $r$  hyperbolic differential system, as defined in [8], is a decomposable differential system of type  $[2, 2]$ . The associated characteristic systems defined in [8] coincide with the singular Pfaffian systems (4.4).

The definition of a Darboux integrable differential system is given in terms of its singular Pfaffian systems.

**Definition 4.2.** *A pair of Pfaffian systems  $\hat{V}$  and  $\check{V}$  on a manifold  $M$  define a **Darboux pair** if*

$$\text{[i]} \quad \hat{V} + \check{V}^\infty = T^*M \quad \text{and} \quad \check{V} + \hat{V}^\infty = T^*M, \quad \text{and} \quad (4.6)$$

$$\text{[ii]} \quad \hat{V}^\infty \cap \check{V}^\infty = \{0\}. \quad (4.7)$$

*A decomposable differential system  $\mathcal{I}$  is **Darboux integrable** if its the singular Pfaffian systems (4.4) define a Darboux pair.*

This definition of Darboux integrability is slightly more general than that given in [3], where it was assumed that the singular systems  $\hat{V}$  and  $\check{V}$  are Pfaffian systems. It is a simple matter to argue ([3], Section 2.2) that if  $\mathcal{I}$  is a Darboux integrable, then about each point of  $M$  there exists a local co-frame  $\{\theta, \hat{\eta}, \check{\eta}, \hat{\sigma}, \check{\sigma}\}$  such that

$$I^1 = \text{span}\{\theta, \hat{\eta}, \check{\eta}\} \quad \text{and} \quad \begin{aligned} \hat{V} &= \text{span}\{\theta, \hat{\eta}, \check{\eta}, \hat{\sigma}\}, & \hat{V}^\infty &= \text{span}\{\hat{\eta}, \hat{\sigma}\}, \\ \check{V} &= \text{span}\{\theta, \hat{\eta}, \check{\eta}, \check{\sigma}\}, & \check{V}^\infty &= \text{span}\{\check{\eta}, \check{\sigma}\}. \end{aligned} \quad (4.8)$$

Such co-frames are said to be  ***$\theta$ -adapted co-frames with respect to the Darboux pair***  $\{\hat{V}, \check{V}\}$ .

The next theorem greatly simplifies the task of verifying that a decomposable differential system is Darboux integrable. This simplification will be used in Section 5.1 to prove the first part of Theorem C.

**Theorem 4.3.** *Let  $\mathcal{I}$  be decomposable differential system with singular Pfaffian systems  $\hat{V}$  and  $\check{V}$  and suppose that  $(I^1)^\infty = (\hat{V} \cap \check{V})^\infty = \{0\}$ . If  $\hat{V}$  and  $\check{V}$  satisfy conditions [i] in the definition of a Darboux pair, then condition [ii] is automatically satisfied and  $\mathcal{I}$  is Darboux integrable.*

**Remark 4.4.** The examples we shall study arise from partial differential equations and therefore come naturally with an independence  $\omega^1 \wedge \dots \wedge \omega^m \neq 0$ . In these examples the associated differential systems are decomposable and there exists local co-frames

$$\tilde{\theta}^1, \dots, \tilde{\theta}^r, \hat{\omega}^1, \dots, \hat{\omega}^{m_1}, \hat{\tau}^1, \dots, \hat{\tau}^{p_1}, \check{\omega}^1, \dots, \check{\omega}^{m_2}, \check{\tau}^1, \dots, \check{\tau}^{p_2}$$

with  $m_1 + p_1 \geq 2$ ,  $m_2 + p_2 \geq 2$ ,

$$\omega^1 \wedge \dots \wedge \omega^m = \hat{\omega}^1 \wedge \dots \wedge \hat{\omega}^{m_1} \wedge \check{\omega}^1 \wedge \dots \wedge \check{\omega}^{m_2}, \quad (4.9)$$

and where the two forms  $\hat{\Omega}^a$  and  $\check{\Omega}^\alpha$  now assume the form

$$\hat{\Omega}^a = \hat{L}_{bc}^a \hat{\tau}^b \wedge \hat{\omega}^c \quad \text{and} \quad \check{\Omega}^\alpha = \check{L}_{\beta\gamma}^\alpha \check{\tau}^\beta \wedge \check{\omega}^\gamma. \quad (4.10)$$

In this situation we can re-write the structure equations (4.5) as

$$d\tilde{\theta}^e = A_{ab}^e \hat{\tau}^a \wedge \hat{\omega}^b + B_{\alpha\beta}^e \check{\tau}^\alpha \wedge \check{\omega}^\beta \quad \text{mod } \tilde{\theta}^e. \quad (4.11)$$

■

## 5 Integrable Extensions of Darboux Integrable Differential Systems

In section 5.1 we shall use Theorem 4.3, to prove the first part of Theorem C. Then, in Section 5.2, we introduce a special class of integrable extensions for Darboux integrable systems which we



call *Darboux compatible integrable extensions*. Such extension arise very naturally in the construction of Darboux integral systems by symmetry reduction in Section 6, in the proof of the uniqueness of the local quotient representation in Section 8, and in the analysis Darboux integrable Monge-Ampère equations in Section 11.

### 5.1 Proof of Part [i] of Theorem C

**Theorem 5.1.** *Let  $\mathbf{p} : (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension with  $\mathcal{E} = \mathcal{S}(J) + \mathbf{p}^*(\mathcal{I})$  and  $J$  an admissible sub-bundle of  $T^*N$ .*

[i] *If  $\mathcal{I}$  is decomposable of type  $[p, \rho]$  with singular systems  $\hat{\mathcal{V}}$  and  $\check{\mathcal{V}}$ , then  $\mathcal{E}$  is decomposable of type  $[p, \rho]$  with singular systems*

$$\hat{\mathcal{Z}} = \mathcal{S}(J) + \mathbf{p}^*(\hat{\mathcal{V}}) \quad \text{and} \quad \check{\mathcal{Z}} = \mathcal{S}(J) + \mathbf{p}^*(\check{\mathcal{V}}). \quad (5.1)$$

*The singular Pfaffian systems are*

$$\hat{Z} = J \oplus \mathbf{p}^*(\hat{V}) \quad \text{and} \quad \check{Z} = J \oplus \mathbf{p}^*(\check{V}). \quad (5.2)$$

[ii] *If  $\mathcal{I}$  is Darboux integrable, and  $(\mathcal{E}^1)^\infty = 0^1$ , then the integrable extension  $\mathcal{E}$  is Darboux integrable.*

*Proof.* Let  $\{\tilde{\theta}^e, \hat{\sigma}^a, \check{\sigma}^\alpha\}$  be a local co-frame on  $U_0 \subset M$  and let  $\hat{\Omega}^c, \check{\Omega}^\gamma$  be two-forms on  $U_0$  such that (see equation (4.2))

$$\mathcal{I}|_U = \langle \tilde{\theta}^e, \hat{\Omega}^c, \check{\Omega}^\gamma \rangle_{\text{alg}}.$$

The structure equations for the 1-forms  $\tilde{\theta}^e$  are (see (4.5))

$$d\tilde{\theta}^e = A_e^e \hat{\Omega}^c + B_\gamma^e \check{\Omega}^\gamma \quad \text{mod } \{ \tilde{\theta}^e \}, \quad (5.3)$$

and the singular systems are (see (4.3))

$$\hat{\mathcal{V}}|_U = \langle \tilde{\theta}^e, \hat{\sigma}^a, \check{\Omega}^\gamma \rangle_{\text{alg}}. \quad \text{and} \quad \check{\mathcal{V}}|_U = \langle \tilde{\theta}^e, \check{\sigma}^\alpha, \hat{\Omega}^c \rangle_{\text{alg}}.$$

Now choose an open set  $U \subset \mathbf{p}^{-1}(U_0)$  and a local basis of sections  $\xi^u$  for  $J$ . Allowing for a slight abuse of notation, we have that the 1-forms  $\{\tilde{\theta}^e, \xi^u, \hat{\sigma}^a, \check{\sigma}^\alpha\}$  are a local co-frame on  $U$ . By the integrable extension property (2.3) the structure equations for  $\mathcal{E}$  are (5.3) and

$$d\xi^u = F_v^u \wedge \xi^v + G_e^u \wedge \tilde{\theta}^e + H_e^u \hat{\Omega}^c + L_\gamma^u \check{\Omega}^\gamma, \quad (5.4)$$

where the  $F_v^u, G_e^u$  are 1-forms on  $U$  and  $H_b^u, L_\beta^u \in C^\infty(U)$ . On account of these structure equations, we have that

$$\mathcal{E}|_U = \langle \tilde{\theta}^e, \xi^u, \hat{\Omega}^b, \check{\Omega}^\beta \rangle_{\text{alg}} \quad (5.5)$$

---

<sup>1</sup>See remark **IE** [vi] on page 4.

and  $\mathcal{E}$  is clearly decomposable of type  $[p, \rho]$  with the singular systems

$$\hat{Z}|_U = \langle \tilde{\theta}^e, \xi^u, \hat{\sigma}^a, \check{\Omega}^\gamma \rangle_{\text{alg}}, \quad \text{and} \quad \check{Z}|_U = \langle \tilde{\theta}^e, \xi^u, \check{\sigma}^\alpha, \hat{\Omega}^c \rangle_{\text{alg}}. \quad (5.6)$$

The corresponding singular Pfaffian systems are

$$\hat{Z}|_U = \{ \tilde{\theta}^e, \xi^u, \hat{\sigma}^a \} \quad \text{and} \quad \check{Z}|_U = \{ \tilde{\theta}^e, \xi^u, \check{\sigma}^\alpha \}. \quad (5.7)$$

Equations (5.6) and (5.7) prove (5.1) and (5.2).

To prove part [ii], suppose that  $\mathcal{I}$  is Darboux integrable in which case we have  $\hat{V} + \check{V}^\infty = T^*M$ . By equation (5.2) we have  $\mathbf{p}^*(\check{V}^\infty) \subset \check{Z}^\infty$  and therefore, on account of the hypothesis  $\hat{V} + \check{V}^\infty = T^*M$  and (2.4) we deduce that

$$\hat{Z} + \check{Z}^\infty \supset J + \mathbf{p}^*(\hat{V}) + \mathbf{p}^*(\check{V}^\infty) = J \oplus \mathbf{p}^*(\hat{V} + \check{V}^\infty) = J \oplus \mathbf{p}^*(T^*M) = T^*N.$$

This proves that the first part of condition [i] in equation (4.6) is satisfied. The second equation is similarly proved. Theorem 4.3 then shows  $\hat{Z}$  and  $\check{Z}$  define a Darboux pair, and so  $\mathcal{E}$  is Darboux integrable.  $\blacksquare$

We remark that if  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  is an integrable extension of decomposable systems  $\mathcal{E}$  and  $\mathcal{I}$ , then the relations (5.2) between the singular systems for  $\mathcal{E}$  and  $\mathcal{I}$  need not automatically hold.

## 5.2 A special class of integrable extensions for Darboux integrable differential systems

Theorem 5.1 shows that the property of Darboux integrability is preserved under integrable extension but without further assumptions one cannot in general determine the number of Darboux invariants for the extension in terms of the number of Darboux invariants of the base differential system. This, in turn, implies that one cannot, in any meaningful way, characterize the Vessiot algebra (the fundamental algebraic invariant of a Darboux integrable system, see Section 7) of the extension in terms of the Vessiot algebra of the base differential system.

In this section we introduce a class of integrable extensions  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  for which the geometric properties of  $\mathcal{E}$  and  $\mathcal{I}$ , in regards to their Darboux integrability, are tightly related.

**Definition 5.2.** Let  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension of decomposable systems  $\mathcal{E}$  and  $\mathcal{I}$  with singular Pfaffian systems  $\{\hat{Z}, \check{Z}\}$  and  $\{\hat{V}, \check{V}\}$  respectively.

[i] The extension  $(\mathcal{E}, \mathcal{I})$  is said to be **compatible with respect to the singular Pfaffian systems**  $\{\hat{Z}, \check{Z}\}$  and  $\{\hat{V}, \check{V}\}$  if there is an admissible sub-bundle  $J \subset T^*N$  (see (2.2)) such that

$$\hat{Z} = J \oplus \mathbf{p}^*(\hat{V}) \quad \text{and} \quad \check{Z} = J \oplus \mathbf{p}^*(\check{V}). \quad (5.8)$$

[ii] The extension  $(\mathcal{E}, \mathcal{I})$  is **Darboux compatible** if there exists a sub-bundle  $\hat{J} \subset \hat{Z}^\infty \cap \check{Z}$  which is simultaneously an admissible sub-bundle for the 3 integrable extensions  $\mathbf{p}: \mathcal{E} \rightarrow \mathcal{I}$ ,  $\mathbf{p}: \hat{Z} \rightarrow \check{V}$  and  $\mathbf{p}: \check{Z}^\infty \rightarrow \check{V}^\infty$ , that is,

$$\mathcal{E} = \mathcal{S}(\hat{J}) + \mathbf{p}^*(\mathcal{I}), \quad \check{Z} = \mathcal{S}(\hat{J}) + \mathbf{p}^*(\check{V}), \quad \hat{Z}^\infty = \hat{J} \oplus \mathbf{p}^*(\hat{V}^\infty), \quad (5.9)$$

and, similarly, a sub-bundle  $\check{J} \subset \check{Z}^\infty \cap \hat{Z}$  such that

$$\mathcal{E} = \mathcal{S}(\check{J}) + \mathbf{p}^*(\mathcal{I}), \quad \hat{Z} = \mathcal{S}(\check{J}) + \mathbf{p}^*(\hat{V}), \quad \check{Z}^\infty = \check{J} \oplus \mathbf{p}^*(\check{V}^\infty). \quad (5.10)$$

The last equations in (5.9) and (5.10) state that the number of Darboux invariants for each singular Pfaffian system of  $\mathcal{E}$  is  $\text{rank}(J)$  more than those for the corresponding Pfaffian systems of  $\mathcal{I}$ . In terms of this definition, Theorem 5.1 states that if  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  is an integrable extension and  $\mathcal{I}$  is decomposable, then there always exists singular Pfaffian systems for  $\mathcal{E}$  which are compatible. We shall see in Theorem 6.4 that Darboux compatible extensions naturally arise when integrable extensions are constructed by group reductions using diagonal actions.

**Remark 5.3.** Darboux compatibility implies that if  $\phi: P \rightarrow N$  is an (immersed) integral manifold for  $\hat{Z}^\infty$ , then  $\mathbf{p} \circ \phi$  is an integral manifold for  $\hat{V}^\infty$ . Moreover, if  $(\phi, P)$  is of maximum dimension for  $\hat{Z}^\infty$  then, by **IE [v]** in Section 2,  $(\mathbf{p} \circ \phi, P)$  is of (the same) maximal dimension for  $\hat{V}^\infty$ .

If  $s_1: P_1 \rightarrow N$  is an integral manifold of  $\hat{Z}^\infty$  of maximal dimension, then  $s_1^*(\hat{J}) = 0$ . By virtue of this observation and the second equation in (5.9), we conclude that

$$s_1^*(\hat{Z}) = s_1^*(\mathcal{S}(\hat{J}) + \mathbf{p}^*(\hat{V})) = s_1^* \circ \mathbf{p}^*(\hat{V}). \quad (5.11)$$

The next theorem gives alternative criteria for an integrable extension to be Darboux compatible. This result will be used in Sections 6 and 11.

**Theorem 5.4.** Let  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension of decomposable systems  $\mathcal{E}$  and  $\mathcal{I}$  with the compatible singular systems  $\{\hat{Z}, \check{Z}\}$  and  $\{\hat{V}, \check{V}\}$  respectively. Assume that  $\mathcal{E}$  is Darboux integrable. Then  $(\mathcal{E}, \mathcal{I})$  is Darboux compatible if and only if

$$\begin{aligned} \text{[i]} \quad & \ker(\mathbf{p}_*) \cap \text{ann}(\hat{Z}^\infty) = 0, \quad \ker(\mathbf{p}_*) \cap \text{ann}(\check{Z}^\infty) = 0 \quad \text{and} \\ \text{[ii]} \quad & \text{rank}(\hat{Z}^\infty) = \text{rank}(\ker(\mathbf{p}_*)) + \text{rank}(\hat{V}^\infty), \quad \text{rank}(\check{Z}^\infty) = \text{rank}(\ker(\mathbf{p}_*)) + \text{rank}(\check{V}^\infty). \end{aligned}$$

The rank equalities [ii] imply that the number of independent Darboux invariants for  $\hat{Z}$  and  $\check{Z}$  is increased from the number of independent Darboux invariants for  $\hat{V}$  and  $\check{V}$  by exactly the fiber dimension of the submersion  $\mathbf{p}$ .

*Proof.* It is easy to check that (5.9) and (5.10) implies conditions [i] and [ii] of Theorem 5.4. To prove the converse, we shall need the following rank equalities

$$\text{rank}(\hat{Z}^\infty \cap \check{Z}) = \text{rank}(\ker \mathbf{p}_*) + \text{rank}(\hat{V}^\infty \cap \check{V}) \quad \text{and} \quad (5.12)$$

$$\text{rank}(\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*)) = \text{rank}(\hat{Z}^\infty) - \text{rank}(\ker \mathbf{p}_*) \quad (5.13)$$

which we now derive from the hypothesis of Theorem 5.4.

Since  $\mathcal{E}$  is Darboux integrable, (4.6) implies that  $\hat{Z}^\infty + \check{Z} = T^*N$  and thus

$$\dim N = \text{rank } \hat{Z}^\infty + \text{rank } \check{Z} - \text{rank}(\hat{Z}^\infty \cap \check{Z}). \quad (5.14)$$

Therefore, on account of equation (5.14), condition [ii] in Theorem 5.4, equation (4.6) and the definition of integrable extension (to compute  $\dim N$ ), we obtain

$$\begin{aligned} \text{rank}(\hat{Z}^\infty \cap \check{Z}) &= \text{rank } \hat{Z}^\infty + \text{rank } \check{Z} - \dim N \\ &= [\text{rank}(\ker \mathbf{p}_*) + \text{rank } \hat{V}^\infty] + [\text{rank}(\ker \mathbf{p}_*) + \text{rank } \check{V}] - [\text{rank}(\ker \mathbf{p}_*) + \dim M] \\ &= \text{rank}(\ker(\mathbf{p}_*)) + \text{rank } \hat{V}^\infty + \text{rank } \check{V} - \dim M = \text{rank}(\ker \mathbf{p}_*) + \text{rank}(\hat{V}^\infty \cap \check{V}). \end{aligned}$$

Next, we calculate the dimension of the sub-bundle  $\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*)$ , using the transversality condition [i] in Theorem 5.4, to be

$$\begin{aligned} \text{rank}(\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*)) &= \dim(N) - \text{rank}(\text{ann}(\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*))) \\ &= \dim(N) - \text{rank}(\text{ann}(\hat{Z}^\infty) + \ker \mathbf{p}_*) = \dim(N) - \text{rank}(\text{ann}(\hat{Z}^\infty)) - \text{rank}(\ker \mathbf{p}_*) \\ &= \text{rank}(\hat{Z}^\infty) - \text{rank}(\ker \mathbf{p}_*). \end{aligned}$$

Equations (5.12) and (5.13) are now established. The rank conditions in [ii] then give

$$\text{rank}(\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*)) = \text{rank}(\hat{V}^\infty) = \text{rank}(\mathbf{p}^*(\hat{V}^\infty)). \quad (5.15)$$

The inclusions (see (5.8))

$$\mathbf{p}^*(\hat{V}) \subset \hat{Z}, \quad \mathbf{p}^*(\hat{V}^\infty) \subset \hat{Z}^\infty, \quad \mathbf{p}^*(\check{V}) \subset \check{Z}, \quad \mathbf{p}^*(\check{V}^\infty) \subset \check{Z}^\infty \quad (5.16)$$

imply that we can choose sub-bundles  $\hat{J} \subset \hat{Z}^\infty \cap \check{Z}$  and  $\check{J} \subset \check{Z}^\infty \cap \hat{Z}$  such that

$$\hat{Z}^\infty \cap \check{Z} = \hat{J} \oplus \mathbf{p}^*(\hat{V}^\infty \cap \check{V}) \quad \text{and} \quad \check{Z}^\infty \cap \hat{Z} = \check{J} \oplus \mathbf{p}^*(\check{V}^\infty \cap \hat{V}). \quad (5.17)$$

Equation (5.12) (and the above formula for  $\hat{Z}^\infty \cap \check{Z}$ ) imply that

$$\text{rank } \hat{J} = \text{rank } \check{J} = \text{rank}(\ker \mathbf{p}_*). \quad (5.18)$$

We claim that the bundles  $\hat{J}$  and  $\check{J}$  satisfy (5.9) and (5.10).

The next step in the proof is to check that these complementary bundles  $\hat{J}$  and  $\check{J}$  are transverse to  $\mathbf{p}$ . We begin by noting that the inclusions (5.16) and rank equality (5.15) imply that

$$\hat{Z}^\infty \cap \text{ann}(\ker \mathbf{p}_*) = \mathbf{p}^*(\hat{V}^\infty). \quad (5.19)$$

Since  $\text{rank}(\hat{J}) = \text{rank}(\ker \mathbf{p}_*)$ , transversality is equivalent to the non-degeneracy of the canonical pairing, restricted to  $(\ker \mathbf{p}_*) \times \hat{J}$ . Let  $\alpha \in \hat{J}$  and suppose  $\alpha(X) = 0$  for all  $X \in \ker \mathbf{p}_*$ , that is,

suppose  $\alpha \in \hat{J}_{\mathbf{p},\mathbf{sb}}$ . Since  $\hat{J} \subset \hat{Z}^\infty$  and  $\hat{J} \subset \check{Z}$  we have  $\alpha \in \hat{Z}_{\mathbf{p},\mathbf{sb}}^\infty$  and  $\alpha \in \check{Z}_{\mathbf{p},\mathbf{sb}}$ . Equation (5.19) implies  $\alpha \in \mathbf{p}^*(\hat{V}^\infty)$  while equation (5.2) implies  $\alpha \in \mathbf{p}^*(\check{V})$ . Therefore  $\alpha \in \mathbf{p}^*(\hat{V}^\infty \cap \check{V})$ , and so by (5.17),  $\alpha = 0$ .

Transversality shows that  $\hat{J} \cap \mathbf{p}^*(\check{V}) = \hat{J} \cap \mathbf{p}^*(\hat{V}^\infty) = \{0\}$  and therefore, by dimensional considerations,

$$\check{Z} = \hat{J} \oplus \mathbf{p}^*(\check{V}), \quad \hat{Z}^\infty = \hat{J} \oplus \mathbf{p}^*(\hat{V}^\infty), \quad \hat{Z} = \check{J} \oplus \mathbf{p}^*(\hat{V}), \quad \check{Z}^\infty = \check{J} \oplus \mathbf{p}^*(\check{V}^\infty).$$

Finally, from the definition of Darboux pair, we have  $\hat{Z}^\infty \cap \check{Z} \subset E^1$  and  $\check{Z}^\infty \cap \hat{Z} \subset E^1$  and therefore  $\hat{J} \subset E^1$  and  $\check{J} \subset E^1$ . The assumption that the singular systems  $\{\hat{Z}, \check{Z}\}$  and  $\{\hat{V}, \check{V}\}$  are compatible implies that

$$\text{rank } E^1 - \text{rank } \mathbf{p}^*(I^1) = \text{rank } \hat{Z} - \text{rank } \mathbf{p}^*(\hat{V}) = \text{rank } \check{Z} - \text{rank } \mathbf{p}^*(\check{V}). \quad (5.20)$$

Thus, again by dimensional considerations,

$$E^1 = \hat{J} \oplus \mathbf{p}^*(I^1) = \check{J} \oplus \mathbf{p}^*(I^1) \quad (5.21)$$

and the proof is complete. ■

## 6 Group Theoretic Constructions of Darboux Integrable Systems

Let  $G$  be a Lie group acting on manifolds  $M_1$  and  $M_2$ . In Section 3 of [3] a general group theoretic construction of Darboux integrable systems was given, based upon symmetry reduction with respect to the diagonal action  $G_{\text{diag}}$  on the product manifold  $M_1 \times M_2$ . In this section we extend this result to the more general case of symmetry reduction by subgroups  $L$  of the product group  $G \times G$ . We also show, in the special case of the diagonal action  $G_{\text{diag}}$ , that symmetry reduction leads to pairs of Darboux integrable systems which are always Darboux compatible in the sense described in Section 5.2.

To begin, let  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be EDS on manifolds  $M_1$  and  $M_2$ . Then the *direct sum*  $\mathcal{K}_1 + \mathcal{K}_2$  is the EDS on  $M_1 \times M_2$  which is algebraically generated by the pullbacks of  $\mathcal{K}_1$  and  $\mathcal{K}_2$  to  $M_1 \times M_2$  by the canonical projection maps  $\pi_a: M_1 \times M_2 \rightarrow M_a$ . The system  $\mathcal{K}_1 + \mathcal{K}_2$  is clearly decomposable with singular Pfaffian systems

$$\hat{W} = K_1^1 \oplus T^*M_2 \quad \text{and} \quad \check{W} = T^*M_1 \oplus K_2^1. \quad (6.1)$$

It is a simple matter to check that  $\{\hat{W}, \check{W}\}$  form a Darboux pair if and only if  $(K_a^1)^\infty = 0$ . Let  $L$  be a subgroup of the product group  $G \times G$ , let  $\rho_a: G \times G \rightarrow G$ ,  $a = 1, 2$  be the projection onto the

$a^{\text{th}}$  factor and let  $L_a = \rho_a(L) \subset G$ . The action of  $L$  on  $M_1 \times M_2$  is then given in terms of these projection maps and the actions of  $G$  on  $M_1$  and  $M_2$  by

$$\ell \cdot (x_1, x_2) = (\rho_1(\ell) \cdot x_1, \rho_2(\ell) \cdot x_2) \quad \text{for } \ell \in L. \quad (6.2)$$

The projection maps  $\pi_a : M_1 \times M_2 \rightarrow M_a$  are equivariant with respect to the actions of  $L$  and  $L_a$ , that is, for any  $\ell \in L$  and  $x \in M_1 \times M_2$

$$\pi_a(\ell \cdot x) = \rho_a(\ell) \cdot \pi_a(x). \quad (6.3)$$

In the special case of the diagonal action  $G_{\text{diag}}$ , each  $L_a \cong G$  and the actions of  $L_a$  on  $M_a$  coincide with the original actions of  $G$  on  $M_a$ . No assumptions will be made regarding the dimensions of the Lie groups  $L_a$ .

We also suppose that  $L$  acts regularly on  $M_1 \times M_2$  and let  $\mathbf{q}_L : M_1 \times M_2 \rightarrow (M_1 \times M_2)/L$  be the canonical submersion. For a given point  $(p_1, p_2) \in M_1 \times M_2$ , let  $\iota_{M_1} : M_1 \rightarrow M_1 \times M_2$  and  $\iota_{M_2} : M_2 \rightarrow M_1 \times M_2$  be the inclusion maps.

$$\iota_{M_1}(x_1) = (x_1, p_2) \quad \text{and} \quad \iota_{M_2}(x_2) = (p_1, x_2) \quad \text{and set} \quad (6.4)$$

$$\mathbf{q}_{M_a} = \mathbf{q}_L \circ \iota_{M_a} : M_a \rightarrow (M_1 \times M_2)/L$$

Let  $\mathbf{q}_{L_a} : M_a \rightarrow M_a/L_a$  be the canonical quotient maps. By the  $L$ -equivariance of the projection maps  $\pi_a$  (see (6.3)), we can then define maps

$$\mathbf{p}_a : (M_1 \times M_2)/L \rightarrow M_a/L_a \quad (6.5)$$

such that the diagrams

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\mathbf{q}_L} & (M_1 \times M_2)/L \\ \pi_1 \downarrow & & \downarrow \mathbf{p}_1 \\ M_1 & \xrightarrow{\mathbf{q}_{L_1}} & M_1/L_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\mathbf{q}_L} & (M_1 \times M_2)/L \\ \pi_2 \downarrow & & \downarrow \mathbf{p}_2 \\ M_2 & \xrightarrow{\mathbf{q}_{L_2}} & M_2/L_2 \end{array} \quad (6.6)$$

commute. If we make the slightly stronger assumptions that the groups  $L_a$  act regularly on  $M_a$ , then the maps  $\mathbf{q}_{L_a}$  and  $\mathbf{q}_{L_a} \circ \pi_a$  are smooth submersions and hence, by Theorem 3.1, the maps  $\mathbf{p}_a$  are smooth submersions.

The following theorem summarizes the essential facts regarding the construction of Darboux integrable systems by symmetry reduction.

**Theorem 6.1.** *Let  $\mathcal{K}_a, a = 1, 2$  be exterior differential systems on  $M_a, a = 1, 2$  and let*

$$\hat{W} = K_1^1 + T^*M_2, \quad \check{W} = T^*M_1 + K_2^1 \quad (6.7)$$

*be the corresponding Darboux pair on  $M_1 \times M_2$ . Assume that  $(K_a^1)^\infty = 0$ . Consider a Lie group  $G$  which acts freely on  $M_a$ , is a common symmetry group of both  $\mathcal{K}_1$  and  $\mathcal{K}_2$ , and acts transversely to*

$\mathcal{K}_1$  and  $\mathcal{K}_2$ . Assume also that the actions of  $L \subset G \times G$  on  $M_1 \times M_2$  and  $L_a$  on  $M_a$  are regular and set

$$M = (M_1 \times M_2)/L, \quad \hat{V} = (K_1^1 + T^*M_2)/L, \quad \check{V} = (T^*M_1 + K_2^1)/L. \quad (6.8)$$

Finally, assume that  $\hat{V}^\infty$  and  $\check{V}^\infty$  are constant rank bundles.

[i] Then  $\{\hat{V}, \check{V}\}$  define a Darboux pair on  $M$  with <sup>2</sup>

$$\hat{V}^\infty = (0 + T^*M_2)/L = \hat{W}^\infty/L, \quad \check{V}^\infty = (T^*M_1 + 0)/L = \check{W}^\infty/L \quad \text{and} \quad (6.9)$$

$$\dim(\hat{V}^\infty) = \dim M_2 - \dim L_2, \quad \dim(\check{V}^\infty) = \dim M_1 - \dim L_1. \quad (6.10)$$

[ii] The quotient differential system  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/L$  on  $M$  is Darboux integrable with singular Pfaffian systems (6.8).

[iii] The bundles  $\hat{V}^\infty$  and  $\check{V}^\infty$  are given by

$$\hat{V}^\infty = \mathbf{p}_2^*(T^*(M_2/L_2)), \quad \text{and} \quad \check{V}^\infty = \mathbf{p}_1^*(T^*(M_1/L_1)). \quad (6.11)$$

For parts [iv] and [v], we restrict to the special case where  $L = G_{\text{diag}}$ .

[iv] The quotient map  $\mathbf{q}_{G_{\text{diag}}} : (\mathcal{K}_1 \times \mathcal{K}_2, M_1 \times M_2) \rightarrow (\mathcal{I}, M)$  defines a Darboux compatible integrable extension with respect to the singular Pfaffian systems (6.7) and (6.8).

[v] Assume that  $M_1$  and  $M_2$  are connected and that  $G$  acts regularly on each manifold  $M_a$ . Then the maps (see (6.4))

$$\mathbf{q}_{M_1} : M_1 \rightarrow M \quad \text{and} \quad \mathbf{q}_{M_2} : M_2 \rightarrow M \quad (6.12)$$

define imbedded, maximal integral manifolds for  $\hat{V}^\infty$  and  $\check{V}^\infty$ .

We prove all five parts of this theorem for the special case of the diagonal action  $G_{\text{diag}}$  in Section 6.1. The generalizations of parts [i] – [iii] for non-diagonal actions is shown to reduce to the diagonal case in Section 6.2. To prove [iv] and [v] we will make repeated use of the simple observation that for free, diagonal group actions the distribution  $\mathbf{\Gamma}_{G_{\text{diag}}}$  of infinitesimal generators for the action of  $G_{\text{diag}}$  satisfies

$$\mathbf{\Gamma}_{G_{\text{diag}}} \cap (TM_1 + 0) = \mathbf{\Gamma}_{G_{\text{diag}}} \cap (0 + TM_2) = \{0\}. \quad (6.13)$$

For non-diagonal actions equations (6.13) fail to hold and statements [iv] and [v] are not true.

**Remark 6.2.** By definition, the Darboux invariants for the Darboux pair  $\{\hat{V}, \check{V}\}$  are the first integrals for  $\hat{V}^\infty$  or  $\check{V}^\infty$ , that is,  $C^\infty(M)$  functions  $f$  such that  $df \in \hat{V}^\infty$  or  $df \in \check{V}^\infty$ . Equations (6.11) clearly imply that if  $g$  is smooth function on  $M_2/L_2$ , then  $g \circ \mathbf{p}_2$  is a first integral for  $\hat{V}^\infty$ . Of course, smooth functions on  $M_2/L_2$  are in one-to-one correspondence with  $L_2$ -invariant functions on  $M_2$ . Accordingly, there is a one-to-one correspondence between the Darboux invariants for the Darboux pair  $\{\hat{V}, \check{V}\}$  on  $(M_1 \times M_2)/G$  and the  $L_a$ -invariant functions on  $M_a$ .

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<sup>2</sup>The notation  $0 + TM_2^*$  indicates that this bundle is to be viewed as sub-bundle of  $T^*M_1 + T^*M_2$ .

**Remark 6.3.** Note that  $G_{\text{diag}}$  is a closed sub-group of  $G \times G$ . Therefore, if  $G$  acts freely and regularly on  $M_1$  and  $M_2$  then, by Remark 3.3, the action of  $G_{\text{diag}}$  is free and regular on  $M_1 \times M_2$ .

## 6.1 Reduction by Diagonal Actions

Part [i] and equation (6.9) of Theorem 6.1 for diagonal actions is proved in [3]. Equation (6.10), in the case of diagonal actions, follows from (6.9), the transversality condition (6.13), (2.13), and the assumption that  $G_{\text{diag}}$  acts freely.

Part [ii] follows immediately from [i]. Here we prove parts [iii], [iv] and [v] (for  $L = G_{\text{diag}}$ ), beginning with part [iv]. Part [iv] follows as a simple corollary to the following more general result which we shall also need in Section 8.

**Theorem 6.4.** *Let  $(\mathcal{K}_1, M_1)$  and  $(\mathcal{K}_2, M_2)$  be exterior differential systems with a common symmetry group  $G$  and with  $(K_a^1)^\infty = 0$ . Let  $H$  be a subgroup of  $G$  and suppose that the actions of  $H$  and  $G$  satisfy all the hypothesis of Theorem 6.1. Then the pair of Darboux integrable systems  $\{\mathcal{E}, \mathcal{I}\}$ , defined by*

$$\mathcal{E} = (\mathcal{K}_1 \times \mathcal{K}_2)/H_{\text{diag}} \quad \text{and} \quad \mathcal{I} = (\mathcal{K}_1 \times \mathcal{K}_2)/G_{\text{diag}} \quad (6.14)$$

*with singular Pfaffian systems as in (6.8), is Darboux compatible with respect to the orbit projection map  $\mathbf{p}: (M_1 \times M_2)/H_{\text{diag}} \rightarrow (M_1 \times M_2)/G_{\text{diag}}$ .*

*Proof.* We check conditions [i] and [ii] of Theorem 5.4. To verify condition [i] we first note that the singular systems for  $\mathcal{E}$  are

$$\hat{Z} = (K_1^1 + T^*M_2)/H_{\text{diag}} \quad \text{and} \quad \check{Z} = (T^*M_1 + K_2^1)/H_{\text{diag}}$$

with

$$\hat{Z}^\infty = (0 + T^*M_2)/H_{\text{diag}} \quad \text{and} \quad \check{Z}^\infty = (T^*M_1 + 0)/H_{\text{diag}}. \quad (6.15)$$

Also, by (3.3), we have

$$\ker(\mathbf{p}_*) = \mathbf{q}_{H_{\text{diag}}*}(\mathbf{\Gamma}_{G_{\text{diag}}}). \quad (6.16)$$

Let  $R \in \ker(\mathbf{p}_*) \cap \text{ann}(\hat{Z}^\infty)$ . Then (6.15) and (6.16) imply there exists vectors  $R_1 \in TM_1 + 0$  and  $R_2 \in \mathbf{\Gamma}_{G_{\text{diag}}}$  such that

$$R = \mathbf{q}_{H_{\text{diag}}*}(R_1) = \mathbf{q}_{H_{\text{diag}}*}(R_2) \quad (6.17)$$

in which case  $R_1 - R_2 \in \mathbf{\Gamma}_{H_{\text{diag}}}$ . Since  $H_{\text{diag}}$  is the restriction of the diagonal action  $G_{\text{diag}}$ , this implies that  $R_1 - R_2 \in \mathbf{\Gamma}_{G_{\text{diag}}}$  and therefore  $R_1 \in \mathbf{\Gamma}_{G_{\text{diag}}}$ . Equation (6.13) then implies that  $R_1 = 0$  and consequently  $R = 0$ . This proves the first equation in [i] in Theorem 5.4 and the proof of the second equation is similar.

To check part [ii] of Theorem 5.4, we note that equations (6.10) give

$$\text{rank}(\hat{Z}^\infty) = \dim M_2 - \dim H \quad \text{and} \quad \text{rank}(\hat{V}^\infty) = \dim M_2 - \dim G. \quad (6.18)$$



Moreover, by equation (6.16), it follows that

$$\text{rank}(\ker(\mathbf{p}_*)) = \dim G - \dim H. \quad (6.19)$$

Equations (6.18) and (6.19) lead to the first equation in condition [ii] of Theorem 5.4. The second equation is similar.  $\blacksquare$

*Proof of Theorem 6.1, Part [iv].* We simply apply Theorem 6.4 with  $H$  as the identity group.  $\blacksquare$

*Proof of Theorem 6.1, Part [iii].* Starting from (6.9), we first use Theorem 2.1 to write

$$\mathbf{q}_{G_{\text{diag}}}^*(\hat{V}^\infty) = (\hat{W}^\infty)_{G_{\text{diag}}, \mathbf{sb}} \quad \text{and} \quad \mathbf{q}_{G_{\text{diag}}}^*(\check{V}^\infty) = (\check{W}^\infty)_{G_{\text{diag}}, \mathbf{sb}}. \quad (6.20)$$

From the assumption that  $(K_a^1)^\infty = 0$ , we immediately conclude that

$$\hat{W}^\infty = 0 + T^*M_2 \quad \text{and} \quad \check{W}^\infty = T^*M_1 + 0 \quad (6.21)$$

from which it then follows that

$$\begin{aligned} (\hat{W}^\infty)_{G_{\text{diag}}, \mathbf{sb}} &= 0 + (T^*M_2)_{G, \mathbf{sb}} = \pi_2^*((T^*M_2)_{G, \mathbf{sb}}) \quad \text{and} \\ (\check{W}^\infty)_{G_{\text{diag}}, \mathbf{sb}} &= (T^*M_1)_{G, \mathbf{sb}} + 0 = \pi_1^*((T^*M_1)_{G, \mathbf{sb}}). \end{aligned} \quad (6.22)$$

Also, the assumption that  $G$  acts regularly on  $M_a$  gives, by (2.12),

$$(T^*M_a)_{G, \mathbf{sb}} = \mathbf{q}_{G_a}^*(T^*(M_a/G)). \quad (6.23)$$

The combination of this equation (for  $a = 2$ ), equation (6.22), and the commutativity of the second diagram in (6.6) yields

$$(\hat{W}^\infty)_{G_{\text{diag}}, \mathbf{sb}} = \pi_2^*((T^*M_2)_{G, \mathbf{sb}}) = \pi_2^*(\mathbf{q}_{G_2}^*(T^*(M_2/G))) = \mathbf{q}_{G_{\text{diag}}}^*(\mathbf{p}_2^*(T^*(M_2/G))). \quad (6.24)$$

This equation, together with (6.20), now leads to the desired result (6.11). The formula for  $\check{V}^\infty$  is similarly established.  $\blacksquare$

*Proof of Theorem 6.1, Part [v].* The fact that the action of  $G$  is free on each  $M_a$  immediately implies that the maps  $\mathbf{q}_{M_a}: M_a \rightarrow M$  are one-to-one. Equation (6.13) shows that  $\ker \mathbf{q}_{M_1, *} = 0$  and  $\ker \mathbf{q}_{M_2, *} = 0$  and therefore the maps  $\mathbf{q}_{M_a}$  are immersions.

Let

$$N_1 = \mathbf{q}_{M_2}(M_2) = \mathbf{q}_{G_{\text{diag}}}(\{p_1\} \times M_2) \quad \text{and} \quad N_2 = \mathbf{q}_{M_1}(M_1) = \mathbf{q}_{G_{\text{diag}}}(M_1 \times \{p_2\}), \quad (6.25)$$

and let  $x_0 = \mathbf{q}_{G_{\text{diag}}}(p_1, p_2)$  and  $[p_a] = \mathbf{q}_{G_a}(p_a)$ . We will show that

$$N_a = \mathbf{p}_a^{-1}([p_a]) \subset (M_1 \times M_2)/G_{\text{diag}}. \quad (6.26)$$

Then, in view of Theorem 6.1, part [iii],  $N_1$  is an integral manifold of  $\check{V}^\infty$  and  $N_2$  is an integral manifold of  $\hat{V}^\infty$ , each through the point  $x_0$ . Since the maps  $\mathbf{p}_a$  are smooth submersions, the level sets  $N_a$  are smooth, imbedded submanifolds of  $M = (M_1 \times M_2)/G_{\text{diag}}$  with

$$\begin{aligned}\dim N_1 &= \dim(M) - \dim(M_1/G) = \dim(M) - \dim(\check{V}^\infty) \quad \text{and} \\ \dim N_2 &= \dim(M) - \dim(M_2/G) = \dim(M) - \dim(\hat{V}^\infty).\end{aligned}$$

This shows that  $N_1$  and  $N_2$  are integral manifolds of maximal dimension. By definition,  $N_1$  and  $N_2$  are connected whenever  $M_2$  and  $M_1$  are connected. To show that  $N_1$  is maximal, let  $\psi : P \rightarrow M$  be a connected integral manifold of  $\hat{V}^\infty$  through  $x_0$ . Then  $(\mathbf{p}_1 \circ \psi)^* = 0$  and so  $\psi(P) \subset \mathbf{p}_1^{-1}([p_1]) \subset N_1$ .

It remains only to check (6.26). We use the commutativity of (6.6) to calculate

$$\mathbf{p}_1^{-1}([p_1]) = \mathbf{q}_{G_{\text{diag}}}((\mathbf{q}_{G_1} \circ \pi_1)^{(-1)}([p_1])) = \mathbf{q}_{G_{\text{diag}}}(\pi_1^{(-1)}(G \cdot p_1)) = \mathbf{q}_{G_{\text{diag}}}((G \cdot p_1) \times M_2). \quad (6.27)$$

But for each  $g \in G$  and  $x_2 \in M_2$  we have

$$(g \cdot p_1, x_2) = g \cdot_{\text{diag}} (p_1, (g^{-1}) \cdot x_2).$$

Therefore  $G \cdot p_1 \times M_2 = G_{\text{diag}}(\{p_1\} \times M_2)$  and consequently, by (6.4) and (6.27)

$$\mathbf{p}_1^{-1}([p_1]) = \mathbf{q}_{G_{\text{diag}}}((G \cdot p_1) \times M_2) = \mathbf{q}_{G_{\text{diag}}}(\{p_1\} \times M_2) = \mathbf{q}_{M_2}(M_2) = N_1. \quad (6.28)$$

The second equation in (6.26) is similarly proved. ■

**Remark 6.5.** Integrable Pfaffian systems whose maximal integral manifolds are all imbedded submanifolds are called *regular integrable Pfaffian systems* [27]. Theorem 6.1, part [v] therefore implies that  $\hat{V}^\infty$  and  $\check{V}^\infty$  are regular integrable Pfaffian systems.

**Remark 6.6.** The hypothesis in part [v] of Theorem 6.1 that the  $M_a$  are connected implies that the inclusions  $\iota_{M_a} : M_a \rightarrow M_1 \times M_2$  are maximal integral manifolds of  $\hat{W}^\infty$  and  $\check{W}^\infty$ . Therefore, by the Darboux compatibility condition established in part [iii] of Theorem 6.1, the manifolds  $N_1$  and  $N_2$  in equation (6.25) are integral manifolds of maximal dimension for  $\hat{V}^\infty$  and  $\check{V}^\infty$ . Part [v] shows, in addition, that these integral manifolds are maximal and imbedded.

## 6.2 Reduction by non-Diagonal Actions

We return to the general case of non-diagonal actions  $L \subset G \times G$  with the objective of proving parts [i]–[iii] of Theorem 6.1 in their full generality. Let

$$A_1 = \rho_1(\ker \rho_2) = \{g_1 \in G \mid (g_1, e) \in L\} \quad \text{and} \quad A_2 = \rho_2(\ker \rho_1) = \{g_2 \in G \mid (e, g_2) \in L\}.$$

These are closed normal subgroups of  $L_1$  and  $L_2$ . The product  $A_1 \times A_2 \subset L$  is therefore also normal and closed. Let  $\tilde{L} = L/(A_1 \times A_2)$ . Then, from the composition of the epimorphisms  $L \rightarrow L_a \rightarrow L_a/A_a$  (with kernel  $A_1 \times A_2$ ) one obtains the isomorphisms

$$\psi_a : \tilde{L} \rightarrow L_a/A_a \quad \text{defined by} \quad \psi_a(\ell A_1 \times A_2) = \rho_a(\ell)A_a \quad \text{for } \ell \in L. \quad (6.29)$$

We assume the groups  $L_a$  act freely and regularly on  $M_a$ . Then, by Remark 3.3, the groups  $A_a$  act freely and regularly on  $M_a$ , and the groups  $L_a/A_a$  act freely and regularly on  $M/A_a$ . Likewise,  $A_1 \times A_2$  acts freely and regularly on  $M_1 \times M_2$ , and the quotient group  $\tilde{L} = L/(A_1 \times A_2)$  acts freely and regularly on  $(M_1 \times M_2)/(A_1 \times A_2)$ . The isomorphism (6.29) implies that  $\tilde{L}$  acts freely and regularly on  $M_1/A_1$  and  $M_2/A_2$ . Remark 6.3 shows that the diagonal action  $\tilde{L}_{\text{diag}}$  on  $M_1/A_1 \times M_2/A_2$  is free and regular.

The following lemma is the key to reducing the non-diagonal versions of parts [i] and [ii] of Theorem 6.1 to the corresponding diagonal versions - it gives a canonical identification of the quotient space  $(M_1 \times M_2)/L$  with a quotient space constructed using the aforementioned diagonal action of  $\tilde{L}$ .

**Lemma 6.7.** *There exists a canonically defined diffeomorphism  $\Phi$  such that the diagram*

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\mathbf{q}_{A_1 \times A_2}} & M_1/A_1 \times M_2/A_2 \\ \mathbf{q}_L \downarrow & & \downarrow \mathbf{q}_{\tilde{L}_{\text{diag}}} \\ (M_1 \times M_2)/L & \xrightarrow{\Phi} & (M_1/A_1 \times M_2/A_2)/\tilde{L}_{\text{diag}} \end{array} \quad (6.30)$$

*commutes.*

*Proof.* The canonical diffeomorphism  $\Phi_2 : (M_1 \times M_2)/(A_1 \times A_2) \rightarrow M_1/A_1 \times M_2/A_2$  is  $\tilde{L} - \tilde{L}_{\text{diag}}$  equivariant and hence induces the right square in the commutative diagram (6.31) (below), where  $\tilde{\Phi}_2$  is a diffeomorphism. Because  $A_1 \times A_2 \subset L$  is normal we may also construct (see Remark 3.3) the left-hand square in (6.31), where  $\tilde{\Phi}_1$  is a diffeomorphism.

$$\begin{array}{ccccc} M_1 \times M_2 & \xrightarrow{\mathbf{q}_{A_1 \times A_2}} & (M_1 \times M_2)/(A_1 \times A_2) & \xrightarrow{\Phi_2} & M_1/A_1 \times M_2/A_2 \\ \mathbf{q}_L \downarrow & & \downarrow \mathbf{q}_{\tilde{L}} & & \downarrow \mathbf{q}_{\tilde{L}_{\text{diag}}} \\ (M_1 \times M_2)/L & \xrightarrow{\tilde{\Phi}_1} & (M_1 \times M_2)/(A_1 \times A_2)/\tilde{L} & \xrightarrow{\tilde{\Phi}_2} & (M_1/A_1 \times M_2/A_2)/\tilde{L}_{\text{diag}} . \end{array} \quad (6.31)$$

This diagram proves the lemma, with  $\Phi = \tilde{\Phi}_2 \circ \tilde{\Phi}_1$ . ■

Our goal is to prove that  $(\mathcal{K}_1 \times \mathcal{K}_2)/L$  is Darboux integrable. We shall do this by first using Theorem 6.1 to prove that  $(\frac{\mathcal{K}_1 \times \mathcal{K}_2}{A_1 \times A_2})/\tilde{L}_{\text{diag}}$  is Darboux integrable and then using the diagram (6.31) to identify these two differential systems.

*Proof of Theorem 6.1, parts [i]–[ii].* Let  $\tilde{\mathcal{K}}_a = \mathcal{K}_a/A_a$  be the reduced differential systems on  $M_a/A_a$ . We shall check that these systems and the actions of  $\tilde{L}$  on  $M_a/A_a$  satisfy all the hypothesis of Theorem 6.1. First, Theorem 2.1 shows that  $(\tilde{K}_a^1)^\infty = 0$ . We have already noted that  $\tilde{L}$  acts freely and regularly on  $M_a/L_a$ . It is easy to check that  $\tilde{L}$  is a symmetry group of  $\tilde{\mathcal{K}}_a$  and is transverse. Then, in accordance with equations (6.8) we set

$$\tilde{M} = (M_1/A_1 \times M_2/A_2)/\tilde{L}_{\text{diag}}, \quad \hat{Z} = (\tilde{K}_1^{-1} + T^*(M_2/A_2))/\tilde{L}_{\text{diag}} \quad \text{and} \quad \check{Z} = (T^*(M_1/A_1) + \tilde{K}_2^1)/\tilde{L}_{\text{diag}}.$$

The last hypothesis of Theorem 6.1 requires that we check that  $\check{Z}^\infty$  and  $\hat{Z}^\infty$  are constant rank bundles. By the definition of the product action of  $A_1 \times A_2$  on  $M_1 \times M_2$  we have

$$\tilde{K}_1^1 + T^*(M_2/A_2) = (K_1^1 + T^*M_2)/(A_1 \times A_2) \quad \text{and} \quad T^*(M_1/A_1) + \tilde{K}_2^1 = (T^*M_1 + K_2^1)/(A_1 \times A_2).$$

Equations (6.8), the application of Theorem 3.1 to the commutative diagram (6.30), and these equations lead to

$$\begin{aligned} \hat{V} &= (K_1^1 + T^*M_2)/L = \Phi^*((K_1^1 + T^*M_2)/(A_1 \times A_2)/\tilde{L}_{\text{diag}}) = \Phi^*(\hat{Z}) \quad \text{and} \\ \check{V} &= (T^*M_1 + K_2^1)/L = \Phi^*((T^*M_1 + K_2^1)/(A_1 \times A_2)/\tilde{L}_{\text{diag}}) = \Phi^*(\check{Z}). \end{aligned} \quad (6.32)$$

The hypothesis that  $\hat{V}^\infty$  and  $\check{V}^\infty$  are constant rank bundles now implies that  $\check{Z}^\infty$  and  $\hat{Z}^\infty$  are constant rank bundles.

The application of Theorem 6.1 part [i], for diagonal actions, then implies that  $\{\hat{Z}, \check{Z}\}$  is a Darboux pair with

$$\hat{Z}^\infty = (0 + T^*(M_2/A_2))/\tilde{L}_{\text{diag}} \quad \text{and} \quad \check{Z}^\infty = (T^*(M_1/A_1) + 0)/\tilde{L}_{\text{diag}}. \quad (6.33)$$

We conclude, again by (6.32), that  $\{\hat{V}, \check{V}\}$  is a Darboux pair with  $\hat{V}^\infty, \check{V}^\infty$  given by (6.9).  $\blacksquare$

*Proof of Theorem 6.1, part [iii].* The counter-parts of the commutative diagrams in (6.6), as applied to the diagonal action of  $\tilde{L}_{\text{diag}}$ , are the commutative diagrams ( $a = 1, 2$ )

$$\begin{array}{ccc} M_1/A_1 \times M_2/A_1 & \xrightarrow{\mathbf{q}_L} & (M_1/A_1 \times M_2/A_2)/\tilde{L}_{\text{diag}} \\ \pi'_a \downarrow & & \downarrow \tilde{\pi}_a \\ M_a/A_a & \xrightarrow{\mathbf{q}_{\tilde{L}}} & (M_a/A_a)/\tilde{L}. \end{array} \quad (6.34)$$

Therefore, by part [iii] of Theorem 6.1 (which we have already verified for diagonal actions) it follows that

$$\hat{Z}^\infty = \tilde{\pi}_2^*(T^*((M_2/A_2)/\tilde{L})) \quad \text{and} \quad \check{Z}^\infty = \tilde{\pi}_1^*(T^*((M_1/A_1)/\tilde{L})). \quad (6.35)$$

Let  $\Psi_a: (M_a/A_a)/\tilde{L} \rightarrow M_a/L_a$  be the canonical smooth diffeomorphisms, let  $\tilde{\mathbf{p}}_a: (M_1/A_1 \times M_2/A_2)/\tilde{L}_{\text{diag}} \rightarrow M_a/L_a$  be the smooth projection maps defined by  $\tilde{\mathbf{p}}_a = \Psi_a \circ \tilde{\pi}_a$  and note, on account of (6.30), that the projection maps  $\mathbf{p}_a: (M_1 \times M_2)/L \rightarrow M_a/L_a$  satisfy  $\mathbf{p}_a = \tilde{\mathbf{p}}_a \circ \Phi$ . Equation (6.35) then yields

$$\hat{Z}^\infty = \tilde{\pi}_2^*(T^*((M_2/A_2)/\tilde{L})) = \tilde{\mathbf{p}}_2^* \circ (\Psi_2^{-1})^*(T^*((M_2/A_2)/\tilde{L})) = \tilde{\mathbf{p}}_2^*(T^*(M_2/L_2))$$

and hence

$$\hat{V}^\infty = \Phi^*\hat{Z}^\infty = \Phi^* \circ \tilde{\mathbf{p}}_2^*(T^*(M_2/L_2)) = \mathbf{p}_2^*(T^*(M_2/L_2)), \quad (6.36)$$

as required. The formula for  $\check{V}^\infty$  is similarly derived.  $\blacksquare$

## 7 The Vessiot Algebras

The fundamental invariant for any Darboux integrable differential system is the Vessiot algebra. In Section 6.1 we recall the definition of this Lie algebra and we give simple necessary conditions under which a mapping between two Darboux integrable systems will induce a Lie algebra monomorphism of Vessiot algebras. These conditions always hold in the special case of Darboux compatible differential systems, as defined in Section 4. In Section 6.2 we calculate the Vessiot algebra for the Darboux integrable systems constructed in Section 5.2. All these results will be used in Section 7 where we characterize those integrable extensions of Darboux integrable equations which arise as group quotients through Theorem A.

### 7.1 The Vessiot Algebra of a Darboux Integrable Differential System

We begin by recalling the fundamental technical result of [3] .

**Theorem 7.1.** *Let  $(\mathcal{I}, M)$  be a Darboux integrable system with singular systems  $\{\hat{V}, \check{V}\}$ . Then there exists, about each point of  $M$ , 0-adapted (see (4.8)) local co-frames  $\{\theta_X, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$  and  $\{\theta_Y, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$  satisfying the structure equations*

$$\begin{aligned} d\hat{\sigma} &= 0, \quad d\hat{\eta} = \hat{\mathbf{A}} \hat{\sigma} \wedge \hat{\sigma} + \hat{\mathbf{G}} \hat{\eta} \wedge \hat{\sigma}, \quad d\check{\sigma} = 0, \quad d\check{\eta} = \check{\mathbf{F}} \check{\sigma} \wedge \check{\sigma} + \check{\mathbf{H}} \check{\eta} \wedge \check{\sigma} \\ d\theta_X &= \frac{1}{2} \tilde{\mathbf{A}} \hat{\pi} \wedge \hat{\pi} + \frac{1}{2} \tilde{\mathbf{B}} \check{\pi} \wedge \check{\pi} + \frac{1}{2} \mathbf{C} \theta_X \wedge \theta_X + \tilde{\mathbf{M}} \hat{\pi} \wedge \theta_X, \quad \text{and} \\ d\theta_Y &= \frac{1}{2} \tilde{\mathbf{E}} \hat{\pi} \wedge \hat{\pi} + \frac{1}{2} \tilde{\mathbf{F}} \check{\pi} \wedge \check{\pi} - \frac{1}{2} \mathbf{C} \theta_Y \wedge \theta_Y + \tilde{\mathbf{N}} \check{\pi} \wedge \theta_Y, \end{aligned} \quad (7.1)$$

where  $\hat{\pi} = (\hat{\sigma}, \hat{\eta})$  and  $\check{\pi} = (\check{\sigma}, \check{\eta})$ . The coefficients  $\mathbf{C} = [C_{ij}^k]$  are constants and the corresponding dual frames  $\{\partial_{\theta_X}, \partial_{\hat{\eta}}, \partial_{\hat{\sigma}}, \partial_{\check{\eta}}, \partial_{\check{\sigma}}\}$  and  $\{\partial_{\theta_Y}, \partial_{\hat{\eta}}, \partial_{\hat{\sigma}}, \partial_{\check{\eta}}, \partial_{\check{\sigma}}\}$  satisfy

$$[\partial_{\theta_{X_i}}, \partial_{\theta_{Y_j}}] = 0. \quad (7.2)$$

Equations (7.1) imply that

$$[\partial_{\theta_{X_i}}, \partial_{\theta_{X_j}}] = -C_{ij}^k \partial_{\theta_{X_k}}, \quad [\partial_{\theta_{Y_i}}, \partial_{\theta_{Y_j}}] = C_{ij}^k \partial_{\theta_{Y_k}}. \quad (7.3)$$

and therefore the constants  $\mathbf{C}$  are the structure constants for a real Lie algebra. Any pair of 0-adapted co-frames satisfying (7.1) and (7.2) are said to be **4-adapted**.

**Remark 7.2.** If  $\{\theta, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$  is any 0-adapted co-frame, then 4-adapted co-frames are constructed by taking  $\theta_X, \theta_Y \in \text{span}\{\theta, \hat{\eta}, \check{\eta}\}$  and keeping the  $\hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}$  unaltered. Consequently, if  $\mathcal{I}$  is Darboux integrable with independence condition (4.9) then the structure equations (7.1) for a 4-adapted co-frames can be written as

$$\begin{aligned} d\hat{\sigma} &= 0, \quad d\hat{\eta} = \hat{\mathbf{A}} \hat{\tau} \wedge \hat{\omega} + \hat{\mathbf{G}} \hat{\eta} \wedge \hat{\sigma}, \quad d\check{\sigma} = 0, \quad d\check{\eta} = \check{\mathbf{F}} \check{\tau} \wedge \check{\omega} + \check{\mathbf{H}} \check{\eta} \wedge \check{\sigma} \\ d\theta_X &= \frac{1}{2} \tilde{\mathbf{A}} \hat{\tau} \wedge \hat{\omega} + \frac{1}{2} \tilde{\mathbf{B}} \check{\tau} \wedge \check{\omega} + \tilde{\mathbf{G}}_1 \hat{\eta} \wedge \hat{\pi} + \tilde{\mathbf{G}}_2 \check{\eta} \wedge \check{\pi} + \frac{1}{2} \mathbf{C} \theta_X \wedge \theta_X + \tilde{\mathbf{M}} \hat{\pi} \wedge \theta_X \\ d\theta_Y &= \frac{1}{2} \tilde{\mathbf{E}} \hat{\tau} \wedge \hat{\omega} + \frac{1}{2} \tilde{\mathbf{F}} \check{\tau} \wedge \check{\omega} + \tilde{\mathbf{H}}_1 \hat{\eta} \wedge \hat{\pi} + \tilde{\mathbf{H}}_2 \check{\eta} \wedge \check{\pi} - \frac{1}{2} \mathbf{C} \theta_Y \wedge \theta_Y + \tilde{\mathbf{N}} \check{\pi} \wedge \theta_Y, \end{aligned} \quad (7.4)$$

where  $\hat{\sigma} = (\hat{\tau}, \hat{\omega})$ , and  $\check{\sigma} = (\check{\tau}, \check{\omega})$ .

Let  $(\mathcal{E}, N)$  be another Darboux integrable differential system with singular Pfaffian systems  $\{\hat{Z}, \check{Z}\}$  and 4-adapted co-frames  $\{\theta'_X, \hat{\eta}', \hat{\sigma}', \check{\eta}', \check{\sigma}'\}$  and  $\{\theta'_Y, \hat{\eta}', \hat{\sigma}', \check{\eta}', \check{\sigma}'\}$ . If  $\phi: N \rightarrow M$  is a smooth constant rank map satisfying  $\phi^*(\mathcal{I}) \subset \mathcal{E}$  and  $\phi^*(\hat{V}) \subset \hat{Z}$  and  $\phi^*(\check{V}) \subset \check{Z}$  then, by virtue of (4.8),

$$\begin{aligned} \phi^*(\{\hat{\eta}, \hat{\sigma}\}) &\subset \{\hat{\eta}', \hat{\sigma}'\}, & \phi^*(\{\theta_X, \hat{\eta}, \check{\eta}, \hat{\sigma}\}) &\subset \{\theta'_X, \hat{\eta}', \check{\eta}', \hat{\sigma}'\}, \\ \phi^*(\{\check{\eta}, \check{\sigma}\}) &\subset \{\check{\eta}', \check{\sigma}'\}, & \phi^*(\{\theta_Y, \hat{\eta}, \check{\eta}, \check{\sigma}\}) &\subset \{\theta'_Y, \hat{\eta}', \check{\eta}', \check{\sigma}'\}. \end{aligned}$$

In particular, there are matrix-valued functions  $\mathbf{R}$  and  $\mathbf{S}$  on  $N$  such that

$$\phi^*(\theta_X) = \mathbf{R}\theta'_X \mod \{\hat{\eta}', \check{\eta}', \hat{\sigma}'\} \quad \text{and} \quad \phi^*(\theta_Y) = \mathbf{S}\theta'_Y \mod \{\hat{\eta}', \check{\eta}', \check{\sigma}'\}. \quad (7.5)$$

The dual vector fields satisfy

$$\phi_*(\partial_{\theta'_X}) = \mathbf{R}\partial_{\theta_X} \quad \text{and} \quad \phi_*(\partial_{\theta'_Y}) = \mathbf{S}\partial_{\theta_Y} \quad (7.6)$$

and, on account of the 4-adapted structure equations, the functions  $\mathbf{R}$  and  $\mathbf{S}$  satisfy

$$\mathbf{R}\mathbf{C}' = \mathbf{C}\mathbf{R}\mathbf{R}, \quad d\mathbf{R} \in \hat{Z}^\infty, \quad \mathbf{S}\mathbf{C}' = \mathbf{S}\mathbf{R}\mathbf{R}, \quad \text{and} \quad d\mathbf{S} \in \check{Z}^\infty. \quad (7.7)$$

These equations prove that  $\phi_{x,*}$ , at each point  $x \in N$ , induces a homomorphism from the Lie algebra of vector-fields  $\{\partial_{\theta'_X}\}$  to the Lie algebra of vector fields  $\{\partial_{\theta_X}\}$ . In particular, when  $\phi$  is a diffeomorphism, this demonstrates that *the Lie algebra defined by the structure constants  $\mathbf{C}$  in a 4-adapted coframe is an invariant of the Darboux integrable system  $\mathcal{I}$* . The corresponding abstract Lie algebra we call the **Vessiot algebra** which we denote by  $\mathbf{vess}(\mathcal{I})$ . We write the Lie algebra homomorphism induced by  $\phi$  as

$$\tilde{\phi}_x: \mathbf{vess}(\mathcal{E}) \rightarrow \mathbf{vess}(\mathcal{I}). \quad (7.8)$$

Equation (4.8) show that the dimension of the Vessiot algebra is

$$\dim \mathbf{vess}(\mathcal{I}) = \dim \text{span}\{\theta_X\} = \dim \text{span}\{\theta_Y\} = \dim M - \text{rank}(\hat{V}^\infty) - \text{rank}(\check{V}^\infty). \quad (7.9)$$

To calculate the Vessiot algebra of a Darboux integrable system, one must calculate a 4-adapted co-frame. Regrettably, there is not at present a more geometric or intrinsic description of this invariant. We have seen that integrable extensions and group quotients of Darboux integrable systems are Darboux integrable but in general it seems difficult to calculate the Vessiot algebra of the extension or quotient differential system in terms of the Vessiot algebra of the original one. We return to this issue in Section 6.2.

**Theorem 7.3.** *Let  $(\mathcal{E}, N)$  and  $\{\mathcal{I}, M\}$  be Darboux integrable differential systems with singular Pfaffian systems  $\{\hat{Z}, \check{Z}\}$  and  $\{\hat{V}, \check{V}\}$  and suppose that  $\phi: N \rightarrow M$  is a smooth map satisfying*

$$\phi^*(\mathcal{I}) \subset \mathcal{E}, \quad \phi^*(\hat{V}) \subset \hat{Z} \quad \text{and} \quad \phi^*(\check{V}) \subset \check{Z}. \quad (7.10)$$

Then the induced homomorphism  $\tilde{\phi}_x: \mathbf{vess}(\mathcal{E}) \rightarrow \mathbf{vess}(\mathcal{I})$  is injective at each point  $x$  if and only if

$$(T^*N)_{\phi, \mathbf{sb}} + (\hat{Z}^\infty \oplus \check{Z}^\infty) = T^*N. \quad (7.11)$$

*Proof.* In order that  $\tilde{\phi}_x$  be injective it is necessary and sufficient that  $\phi_{x,*}$ , restricted to  $\text{span}\{\partial_{\theta'_X}\}$  be injective. Since the co-frame  $\{\theta'_X, \hat{\eta}', \hat{\sigma}', \check{\eta}', \check{\sigma}'\}$  is 0-adapted, we deduce from (4.8) that

$$\text{span}\{\partial_{\theta'_X}\} = \text{ann}(\hat{Z}^\infty \oplus \check{Z}^\infty)$$

and therefore  $\tilde{\phi}_x$  is injective at each point if and only if

$$\ker(\phi_*) \cap \text{ann}(\hat{Z}^\infty \oplus \check{Z}^\infty) = 0. \quad (7.12)$$

The dual of equation (7.12) produces equation (7.11). ■

**Corollary 7.4.** *Let  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension of Darboux systems  $\mathcal{E}$  and  $\mathcal{I}$ . If the pair  $\{\mathcal{E}, \mathcal{I}\}$  is Darboux compatible, then the induced map*

$$\tilde{\mathbf{p}}_x: \mathbf{vess}(\mathcal{E}) \rightarrow \mathbf{vess}(\mathcal{I}) \quad (7.13)$$

*is a Lie algebra monomorphism for each point  $x$ .*

*Proof.* This follows directly from condition [i] of Theorem 5.4 and (7.12). ■

**Theorem 7.5.** *Let  $\mathcal{I}$  be a Darboux integrable involutive Pfaffian system with independence condition (see 4.4). Then the prolongation of  $\mathcal{I}^{[1]}$  is Darboux integrable and the Vessiot algebras  $\mathbf{vess}(\mathcal{I}^{[1]})$  and  $\mathbf{vess}(\mathcal{I})$  are isomorphic.*

*Proof.* The independence condition induces a splitting of the forms  $\hat{\sigma}, \check{\sigma}$  as  $\hat{\sigma} = (\hat{\tau}, \hat{\omega}), \check{\sigma} = (\check{\tau}, \check{\omega})$ , where the forms  $\hat{\tau}, \hat{\omega}, \check{\tau}, \check{\omega}$  can be taken as closed and the independence condition is given in equation (4.9).

Now let  $\{\theta, \hat{\eta}, \check{\eta}, \hat{\sigma}, \check{\sigma}\}$  be a 4-adapted co-frame for  $\mathcal{I}$  on an open set  $U$ , where by equation (7.4) we have

$$\mathcal{I} = \langle \theta, \hat{\eta}, \check{\eta}, \tilde{\mathbf{A}} \hat{\tau} \wedge \hat{\omega}, \tilde{\mathbf{F}} \check{\tau} \wedge \check{\omega}, \tilde{\mathbf{A}} \hat{\tau} \wedge \check{\omega}, \tilde{\mathbf{F}} \check{\tau} \wedge \hat{\omega} \rangle_{\text{alg}}.$$

and

$$\hat{V} = \{\theta, \hat{\eta}, \check{\eta}, \hat{\tau}, \hat{\omega}\}, \quad \check{V} = \{\theta, \hat{\eta}, \check{\eta}, \check{\tau}, \check{\omega}\}.$$

The prolongation of  $\mathcal{I}$  [22], [24] is computed to be

$$\theta_1^a = \hat{\tau}^a - \hat{s}^v \hat{S}_{vc}^a \hat{\omega}^c, \quad \text{and} \quad \theta_2^\alpha = \check{\tau}^\alpha - \check{s}^\nu \check{S}_{\nu\gamma}^\alpha \check{\omega}^\gamma, \quad (7.14)$$

where  $\hat{s}^v \hat{S}_{vc}^a(x)$  and  $\check{s}^\nu \check{S}_{\nu\gamma}^\alpha(x)$  are the general solutions to the linear homogeneous equations (see (4.10))

$$A_{ab}^d \hat{S}_c^a \hat{\omega}^b \wedge \hat{\omega}^c = 0, \quad \text{and} \quad F_{\alpha\beta}^\delta \check{S}_\gamma^\alpha \check{\omega}^\beta \wedge \check{\omega}^\gamma = 0 \quad (7.15)$$

and  $\mathbf{A} = (\tilde{\mathbf{A}}, \hat{\mathbf{A}})$  and  $\mathbf{F} = (\tilde{\mathbf{F}}, \hat{\mathbf{F}})$ . Implicit in the definition of the prolongation of  $\mathcal{I}$  is the assumption that the solutions spaces to these linear systems have constant dimension. The chart for the prolongation space of  $\mathcal{I}^{[1]}$  over  $U$  is  $U^{[1]} = U \times \mathbf{R}[\hat{s}^v] \times \mathbf{R}[\check{s}^\nu]$  with  $\pi : U^{[1]} \rightarrow U$ , and

$$\mathcal{I}^{[1]}|_{U^{[1]}} = \langle \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \rangle_{\text{diff}}. \quad (7.16)$$

In order to show that  $\mathcal{I}^{[1]}$  is decomposable we begin by first taking the exterior derivative of the equations in (7.4) to get

$$d\tilde{\mathbf{A}} \equiv 0 \pmod{\hat{V}}, \quad d\hat{\mathbf{A}} \equiv 0 \pmod{\hat{V}}, \quad d\tilde{\mathbf{F}} \equiv 0 \pmod{\check{V}}, \quad d\hat{\mathbf{F}} \equiv 0 \pmod{\check{V}}.$$

Utilizing this we may solve for  $\hat{S}$  and  $\check{S}$  in equation (7.15) so that

$$d\hat{S}_{vc}^a \equiv 0 \pmod{\hat{V}}, \quad d\check{S}_{\gamma}^\alpha \equiv 0 \pmod{\check{V}}. \quad (7.17)$$

Now using (7.14) and equations (7.4) the structure equations for  $\mathcal{I}^{[1]}$  are

$$\begin{aligned} d\hat{\boldsymbol{\eta}} &\equiv 0 \pmod{I^{[1]}}, \quad d\check{\boldsymbol{\eta}} \equiv 0 \pmod{I^{[1]}}, \quad d\boldsymbol{\theta} \equiv 0 \pmod{I^{[1]}}, \\ d\theta_1^a &= (\hat{S}_{vc}^a d\hat{s}^v + s^v d\hat{S}_{vc}^a) \wedge \hat{\omega}^c, \quad d\check{\theta}_2^\alpha = (\check{S}_{\nu\gamma}^\alpha d\check{s}^\nu + \check{s}^\nu d\check{S}_{\nu\gamma}^\alpha) \wedge \check{\omega}^\gamma. \end{aligned} \quad (7.18)$$

By condition (7.17) the structure equations (7.18) show that system  $\mathcal{I}^{[1]}$  is decomposable with singular Pfaffian system

$$\hat{V}_1 = \{\hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \hat{\omega}, d\hat{s}^v\} \quad \text{and} \quad \check{V}_1 = \{\hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \check{\omega}, d\check{s}^\nu\}. \quad (7.19)$$

Now according to Remark B.3 if  $k_a \hat{S}_{vc}^a = 0$  then  $k_a = 0$  and likewise if  $k_\alpha \check{S}_{\nu\gamma}^\alpha = 0$  then  $k_\alpha = 0$ . The first consequence of this is (or see Lemma B.4)

$$\mathcal{I}^{[1]'}|_{U^{[1]}} = \langle \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta} \rangle_{\text{diff}},$$

from which we may conclude  $\mathcal{I}^{[1]\infty} = 0$ .

The second consequence of this remark allows us to compute the derived systems  $\hat{V}'_1$  and  $\check{V}'_1$ . Using equations (7.18), (7.14), (7.17) the fact that  $\hat{\omega}$  and  $\check{\omega}$  are closed and the above remark, we get

$$\hat{V}'_1 = \{\hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta}, \boldsymbol{\theta}_1, \hat{\omega}, d\hat{s}^v\} = \pi^*(\hat{V}) + \{d\hat{s}^v\} \quad \check{V}'_1 = \{\hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}, \boldsymbol{\theta}, \boldsymbol{\theta}_2, \check{\omega}, d\check{s}^\nu\} = \pi^*(\check{V}) + \{d\check{s}^\nu\} \quad (7.20)$$

from which we see that  $\hat{V}'_1/\pi = \hat{V}$  and  $\check{V}'_1/\pi = \check{V}$ .

It is now a simple matter to use Theorem 4.3 to check that  $\mathcal{I}^{[1]}$  is Darboux integrable. Indeed, equation (7.20) implies that (see also [4])

$$\hat{V}_1^\infty = \pi^*(\hat{V}^\infty) + \{d\hat{s}^v\} = \{\hat{\boldsymbol{\eta}}, \boldsymbol{\theta}_1, \hat{\omega}, d\hat{s}^v\} \quad \text{and} \quad \check{V}_1^\infty = \pi^*(\check{V}^\infty) + \{d\check{s}^\nu\} = \{\check{\boldsymbol{\eta}}, \boldsymbol{\theta}_2, \check{\omega}, d\check{s}^\nu\}. \quad (7.21)$$

Together (7.19) and (7.21) imply that Theorem 4.3 holds and so  $\mathcal{I}^{[1]}$  is Darboux integrable.



To show that  $\mathbf{vess}(\mathcal{I}^{[1]})$  and  $\mathbf{vess}(\mathcal{I})$  are isomorphic we begin noting that by (7.9), equation (7.21) shows that  $\dim \mathbf{vess}(\mathcal{I}^{[1]}) = \dim \mathbf{vess}(\mathcal{I})$ . Moreover, on the open set  $U^{[1]}$

$$T^*U_{\pi, sb}^{[1]} = \{\theta, \hat{\eta}, \check{\eta}, \hat{\tau}, \check{\tau}, \hat{\omega}, \check{\omega}\} \quad (7.22)$$

and  $d\hat{s}^v, d\check{s}^\nu \in \hat{V}_1^\infty \oplus \check{V}_1^\infty$ . Thus  $\pi^*(\mathcal{I}) \subset \mathcal{I}^{[1]}$ ,  $\pi^*(\hat{V}) \subset \hat{V}_1$ ,  $\pi^*(\check{V}) \subset \check{V}_1$  and condition (7.10) of Theorem 7.3 is satisfied. Therefore since  $\mathbf{vess}(\mathcal{I}^{[1]})$  and  $\mathbf{vess}(\mathcal{I})$  are the same dimension, Theorem 7.3 implies they are isomorphic.  $\blacksquare$

**Remark 7.6.** The 4-adapted co-frames for a Darboux integrable system  $\mathcal{I}$  with independence condition naturally lift to give 4-adapted co-frames for  $\mathcal{I}^{[1]}$ . Indeed, the co-frame from above

$$\{\theta, \theta_1, \theta_2, \hat{\eta}, \check{\eta}, \hat{\omega}, \check{\omega}, d\hat{s}^v, d\check{s}^\nu\} \quad (7.23)$$

on the prolongation space  $U^{[1]}$  is already 0-adapted for the Darboux integrable system  $\mathcal{I}^{[1]}$  (see (7.19) and (7.21)) and satisfies

$$\hat{V}_1^\infty \cap \check{V} = \{\theta_1, \hat{\eta}\}, \quad \check{V}^\infty \cap \hat{V} = \{\theta_2, \check{\eta}\}.$$

In this co-frame the structure equations (7.4), treated as part of the structure equations for  $\mathcal{I}^{[1]}$ , become

$$d\theta = \frac{1}{2}\tilde{\mathbf{A}}\hat{\Pi} \wedge \hat{\Pi} + \frac{1}{2}\tilde{\mathbf{B}}\check{\Pi} \wedge \check{\Pi} + \frac{1}{2}\mathbf{C}\theta \wedge \theta + \tilde{\mathbf{M}}_1\theta_1 \wedge \theta + \tilde{\mathbf{M}}_2\hat{\eta} \wedge \theta + \mathbf{M}_3\hat{\omega} \wedge \theta. \quad (7.24)$$

where from equation (7.21) we have  $\hat{\Pi} = \{\theta_1, \hat{\eta}, \hat{\omega}, d\hat{s}^v\}$  as a local basis of sections for  $\hat{V}_1^\infty$ , and  $\check{\Pi} = \{\theta_2, \check{\eta}, \check{\omega}, d\check{s}^\nu\}$ . Equation (7.24) together with (7.18) (using (7.17)) can be written as

$$\begin{aligned} d\theta &= \frac{1}{2}\tilde{\mathbf{A}}\hat{\Pi} \wedge \hat{\Pi} + \frac{1}{2}\tilde{\mathbf{B}}\check{\Pi} \wedge \check{\Pi} + \frac{1}{2}\mathbf{C}\theta \wedge \theta + \tilde{\mathbf{M}}\hat{\Pi} \wedge \theta, \\ d\theta_1 &= \frac{1}{2}\tilde{\mathbf{A}}_1\hat{\Pi} \wedge \hat{\Pi} + \tilde{\mathbf{M}}_1\theta \wedge \hat{\Pi}, \quad d\theta_2 = \frac{1}{2}\tilde{\mathbf{B}}_2\check{\Pi} \wedge \check{\Pi} + \tilde{\mathbf{M}}_2\theta \wedge \hat{\Pi} \end{aligned} \quad (7.25)$$

Therefore the co-frame (7.23) is four-adapted. From equation (7.25) we see directly that the Vessiot algebras  $\mathbf{vess}(\mathcal{I}^{[1]})$  and  $\mathbf{vess}(\mathcal{I})$  are isomorphic.  $\blacksquare$

## 7.2 4-Adapted Coframes and Vessiot Algebras for the Quotient Contruction

In this section we show how to obtain the 4-adapted co-frame for the Darboux integrable system  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}$  directly from the adapted co-frames for each  $\mathcal{K}_a$  introduced in Theorem 2.2. This construction proves that  $\mathbf{vess}(\mathcal{I}) = \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ , and will be needed for the proof of Theorem 8.3. We assume the hypothesis of Theorem 6.1 and, in addition, that the group  $G$  acts regularly on  $M_a$ . We break the construction of these co-frames into a number of steps.

**Step 1.** Fix a point  $p \in M$  and pick points  $p_a \in M_a$  with  $\mathbf{q}_{G_{\text{diag}}}(p_1, p_2) = p$ . Then pick  $G$ -invariant open sets  $U_a \subset M_a$ , with  $p_a \in U_a$ , and open sets  $\bar{U}_a \subset M_a/G$  on which local trivializations

$$\Phi_a : U_a \rightarrow \bar{U}_a \times G, \quad \text{with} \quad \Phi_a = (\mathbf{q}_G^a, \phi_a) \quad \text{and} \quad \phi_a(p_a) = e, \quad (7.26)$$

for  $\mathbf{q}_G^a : M_a \rightarrow M_a/G$  are defined. Let  $\{\boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\sigma}_1\}$  on  $U_1$  and  $\{\boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\sigma}_2\}$  on  $U_2$  be co-frames satisfying conditions [i]-[v] of Theorem 2.2. Then we have

$$K_1^1|_{U_1} = \{\boldsymbol{\theta}_1, \boldsymbol{\eta}_1\} \quad \text{and} \quad K_2^1|_{U_2} = \{\boldsymbol{\theta}_2, \boldsymbol{\eta}_2\} \quad (7.27)$$

and, by parts [ii]-[iii] of Theorem 2.2,

$$\begin{aligned} \boldsymbol{\theta}_1(\mathbf{X}) &= \mathbf{1}, & \boldsymbol{\eta}_1(\mathbf{X}) &= \mathbf{0}, & \boldsymbol{\sigma}_1(\mathbf{X}) &= \mathbf{0}, \\ \boldsymbol{\theta}_2(\mathbf{Y}) &= \mathbf{1}, & \boldsymbol{\eta}_2(\mathbf{Y}) &= \mathbf{0}, & \boldsymbol{\sigma}_2(\mathbf{Y}) &= \mathbf{0}. \end{aligned} \quad (7.28)$$

Here  $\mathbf{X}$  denotes the basis of infinitesimal generators for the action of  $G$  on  $U_1$  and  $\mathbf{Y}$  the basis of infinitesimal generators for  $G$  on  $U_2$ . The structure equations for  $\mathbf{X}$  and  $\mathbf{Y}$  are the same.

For ease of notation we will identify the forms  $\{\boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\sigma}_1\}$  on  $U_1$  with their pullbacks by  $\pi_1$  to  $U_1 \times U_2$  and the forms  $\{\boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\sigma}_2\}$  on  $U_2$  with their pullbacks by  $\pi_2$  to  $U_1 \times U_2$ . Thus

$$\{\boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\sigma}_1, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\sigma}_2\} \quad (7.29)$$

defines a co-frame on  $U_1 \times U_2$ . Let  $\delta : G \times (M_1 \times M_2) \rightarrow M_1 \times M_2$  denote the diagonal action of  $G$  on  $M_1 \times M_2$ , that is,  $\delta_g(x_1, x_2) = (\mu_1(g, x_1), \mu_2(g, x_2))$ . The vector fields  $\mathbf{X} + \mathbf{Y}$  are then a basis for the infinitesimal generators for the diagonal action  $\delta$ . Set  $U = (U_1 \times U_2)/G_{\text{diag}}$ .

**Step 2.** Theorem 2.2, part [ii], states that the forms  $\pi_1 = \{\boldsymbol{\eta}_1, \boldsymbol{\sigma}_1\}$  and  $\pi_2 = \{\boldsymbol{\eta}_2, \boldsymbol{\sigma}_2\}$ , defined on  $U_1$  and  $U_2$  respectively, are  $G$ -basic. Accordingly, we can define forms  $\bar{\boldsymbol{\eta}}_1$  and  $\bar{\boldsymbol{\sigma}}_1$  on  $\bar{U}_1 = U_1/G$  and  $\bar{\boldsymbol{\eta}}_2$  and  $\bar{\boldsymbol{\sigma}}_2$  on  $\bar{U}_2 = U_2/G$  such that

$$\mathbf{q}_G^{1,*}(\bar{\boldsymbol{\eta}}_1) = \boldsymbol{\eta}_1, \quad \mathbf{q}_G^{1,*}(\bar{\boldsymbol{\sigma}}_1) = \boldsymbol{\sigma}_1, \quad \mathbf{q}_G^{2,*}(\bar{\boldsymbol{\eta}}_2) = \boldsymbol{\eta}_2, \quad \mathbf{q}_G^{2,*}(\bar{\boldsymbol{\sigma}}_2) = \boldsymbol{\sigma}_2. \quad (7.30)$$

We then use the maps  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , defined by the commutative diagrams (6.6) (with  $L = G_{\text{diag}}$ ), to define 1-forms  $\hat{\boldsymbol{\pi}} = \{\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\sigma}}\}$  and  $\check{\boldsymbol{\pi}} = \{\check{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}\}$  on  $U$  by

$$\hat{\boldsymbol{\eta}} = \mathbf{p}_2^*(\bar{\boldsymbol{\eta}}_2), \quad \hat{\boldsymbol{\sigma}} = \mathbf{p}_2^*(\bar{\boldsymbol{\sigma}}_2), \quad \check{\boldsymbol{\eta}} = \mathbf{p}_1^*(\bar{\boldsymbol{\eta}}_1), \quad \check{\boldsymbol{\sigma}} = \mathbf{p}_1^*(\bar{\boldsymbol{\sigma}}_1). \quad (7.31)$$

The diagrams (6.6) also show that

$$\mathbf{q}_{G_{\text{diag}}}^*(\hat{\boldsymbol{\eta}}) = \boldsymbol{\eta}_2, \quad \mathbf{q}_{G_{\text{diag}}}^*(\hat{\boldsymbol{\sigma}}) = \boldsymbol{\sigma}_2, \quad \mathbf{q}_{G_{\text{diag}}}^*(\check{\boldsymbol{\eta}}) = \boldsymbol{\eta}_1, \quad \mathbf{q}_{G_{\text{diag}}}^*(\check{\boldsymbol{\sigma}}) = \boldsymbol{\sigma}_1. \quad (7.32)$$

**Step 3.** We now use (2.17) and the trivializations (7.26) to define matrix-valued functions  $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 : U_1 \times U_2 \rightarrow GL(r, \mathbf{R})$  by

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda} \circ \phi_1 \circ \pi_1 \quad \text{and} \quad \boldsymbol{\lambda}_2 = \boldsymbol{\lambda} \circ \phi_2 \circ \pi_2. \quad (7.33)$$

By virtue of the  $G$ -equivariance of the  $\phi_a$ , (2.19) and (7.26) one easily checks that

$$\lambda_{a,j}^i(\delta_g(x_1, x_2)) = \lambda_j^k(g)\lambda_k^i(\phi_a(x_a)) \quad \text{and} \quad \lambda_{a,j}^i(p_1, p_2) = \delta_j^i. \quad (7.34)$$

Next, we use (7.33) to define 1-forms on  $U_1 \times U_2$  by

$$\vartheta_1^i = \lambda_{1,j}^i(\theta_2^j - \theta_1^j) \quad \text{and} \quad \vartheta_2^i = \lambda_{2,j}^i(\theta_2^j - \theta_1^j). \quad (7.35)$$

Clearly, equations (7.28) imply that the forms  $\vartheta_1^i$  and  $\vartheta_2^i$  are  $G_{\text{diag}}$  semi-basic. By part [iv] of Theorem 2.2 and (7.34), it follows that the forms  $\vartheta_a^i$  are  $G$ -invariant and hence  $G$ -basic. Consequently we can define 1-forms  $\theta_X$  and  $\theta_Y$  on  $U$  such that

$$\mathbf{q}_{G_{\text{diag}}}^*(\theta_X) = \vartheta_2 \quad \text{and} \quad \mathbf{q}_{G_{\text{diag}}}^*(\theta_Y) = \vartheta_1. \quad (7.36)$$

For future reference we note, again by (7.34), that  $\vartheta_1^i(p_1, p_2) = \vartheta_2^i(p_1, p_2)$  and hence

$$\theta_X(p) = \theta_Y(p). \quad (7.37)$$

The 1-forms  $\{\theta_\bullet, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$ , where  $\theta_\bullet = \theta_X$  or  $\theta_\bullet = \theta_Y$  define co-frames on  $U$ . We claim that these define the sought after 4-adapted co-frames for  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G$ .

**Step 4.** First we check that the above co-frames are properly aligned with the singular Pfaffian systems  $\{\hat{V}, \check{V}\}$  and their derived flags, in other words, that they define 0-adapted co-frames (see (4.8)). To begin, we immediately deduce, using equations (7.28), that on  $U = U_1 \times U_2$

$$(K_1^1 + T^*M_2)_{G_{\text{diag}}, \text{sb}} = \{\theta_2 - \theta_1, \eta_1, \eta_2, \sigma_2\}, \quad (T^*M_1 + K_2^1)_{G_{\text{diag}}, \text{sb}} = \{\theta_2 - \theta_1, \eta_1, \sigma_1, \eta_2\}, \quad (7.38)$$

$$(T^*M_2)_{G, \text{sb}} = \{\eta_2, \sigma_2\}, \quad (T^*M_1)_{G, \text{sb}} = \{\eta_1, \sigma_1\}, \quad (K_1^1 + K_2^1)_{G_{\text{diag}}, \text{sb}} = \{\theta_2 - \theta_1, \eta_1, \eta_2\}. \quad (7.39)$$

Then, since

$$\begin{aligned} \mathbf{q}_{G_{\text{diag}}}^*\{\theta_\bullet, \hat{\eta}, \check{\eta}, \hat{\sigma}\} &= \{\theta_2 - \theta_1, \eta_1, \eta_2, \sigma_2\} \quad \text{and} \\ \mathbf{q}_{G_{\text{diag}}}^*\{\theta_\bullet, \hat{\eta}, \check{\eta}, \check{\sigma}\} &= \{\theta_2 - \theta_1, \eta_1, \sigma_1, \eta_2 \end{aligned} \quad (7.40)$$

(see (7.32) and (7.36)), it follows from (2.12) and definition (6.8) that

$$\hat{V}|_U = \{\theta_\bullet, \hat{\eta}, \check{\eta}, \hat{\sigma}\} \quad \text{and} \quad \check{V}|_U = \{\theta_\bullet, \hat{\eta}, \check{\eta}, \check{\sigma}\}. \quad (7.41)$$

Moreover, the combination of (7.31), (7.39) and Theorem 6.1[iii] gives

$$\hat{V}^\infty|_U = \mathbf{p}_2^*(T^*(U_2/G)) = \{\hat{\eta}, \hat{\sigma}\} \quad \text{and} \quad \check{V}^\infty|_U = \mathbf{p}_1^*(T^*(U_1/G)) = \{\check{\eta}, \check{\sigma}\}. \quad (7.42)$$

Equations (7.41) and (7.42) show that the co-frames  $\{\theta_\bullet, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$  are 0-adapted co-frames for the Darboux pair  $\{\hat{V}, \check{V}\}$  on  $M$ . We also remark that (see (6.4))

$$\begin{aligned} \mathbf{q}_{M_1}^*(\hat{V}) &= (\iota_{M_1})^*\{\lambda_2(\theta_2 - \theta_1), \eta_1, \eta_2, \sigma_2\} = \{\theta_1, \eta_1\} = K_1^1 \quad \text{and} \\ \mathbf{q}_{M_2}^*(\check{V}) &= (\iota_{M_2})^*\{\lambda_1(\theta_2 - \theta_1), \eta_1, \eta_2, \sigma_2\} = \{\theta_2, \eta_2\} = K_2^1. \end{aligned} \quad (7.43)$$

**Step 5.** We now verify the structure equations in (7.1) The first set of these equations follow directly from (2.23). To prove the remaining structure equations we first recall, from equation (2.23), that the structure equations for the 1-forms  $\theta_1$  and  $\theta_2$  are

$$d\theta_1 = \mathbf{A} \pi_1 \wedge \pi_1 - \frac{1}{2} \mathbf{C} \theta_1 \wedge \theta_1 \quad \text{and} \quad d\theta_2 = \mathbf{B} \pi_2 \wedge \pi_2 - \frac{1}{2} \mathbf{C} \theta_2 \wedge \theta_2. \quad (7.44)$$

The coefficients  $\mathbf{A}$  are defined on  $U_1$ , the coefficients  $\mathbf{B}$  are defined on  $U_2$  and the  $\mathbf{C}$  are the structure constants for the Lie algebra of vector fields  $\mathbf{X}$  (or  $\mathbf{Y}$ ). Secondly, on account of (2.20) and (2.26), we have that

$$d\lambda_{1,j}^i = \pi_1^*(\phi_1^*(d\lambda_j^i)) = \pi_1^*(\phi_1^*(\lambda_k^i C_{lj}^k \tau^j)) = \lambda_{1,k}^j C_{lj}^k \theta_1^j + M_{aj}^i \sigma_1^a.$$

By virtue of (2.23) and this last equation we calculate the structure equations for the forms  $\vartheta_1$  in (7.35) to be

$$\begin{aligned} d\vartheta_1 &= d\lambda_1 \wedge (\theta_2 - \theta_1) + \lambda_1 d(\theta_2 - \theta_1) \\ &= (\lambda_1 \mathbf{C} \theta_1 + \mathbf{M} \pi_1) \wedge (\theta_2 - \theta_1) + \lambda_1 (-\frac{1}{2} \mathbf{C} \theta_2 \wedge \theta_2 + \frac{1}{2} \mathbf{C} \theta_1 \wedge \theta_1 - \mathbf{A} \pi_1 \wedge \pi_1 + \mathbf{B} \pi_2 \wedge \pi_2). \end{aligned}$$

All the terms on the right-hand side of this equation which involve the 2-forms  $\theta_a \wedge \theta_b$  combine to give  $-\frac{1}{2} \mathbf{C} \vartheta_1 \wedge \vartheta_1$  and thus

$$d\vartheta_1 = -\lambda_1 \mathbf{A} \pi_1 \wedge \pi_1 + \lambda_1 \mathbf{B} \pi_2 \wedge \pi_2 - \frac{1}{2} \mathbf{C} \vartheta_1 \wedge \vartheta_1 + \mathbf{M} \lambda_1^{-1} \pi_1 \wedge \vartheta_1. \quad (7.45)$$

A similar equation for  $d\vartheta_2$  holds (except for a change in the sign of the term containing the structure constants  $\mathbf{C}$ ). Finally equations (7.45) (and the counterpart for  $d\vartheta_2$ ) yield the structure equations (7.1).

Note that (7.1) immediately imply that the the dual vector fields  $\partial_{\theta_X^a}$  and  $\partial_{\theta_Y^a}$  satisfy

$$[\partial_{\theta_X^a}, \partial_{\theta_X^b}] = -C_{ab}^e \partial_{\theta_X^e} \quad \text{and} \quad [\partial_{\theta_Y^a}, \partial_{\theta_Y^b}] = C_{ab}^e \partial_{\theta_Y^e}. \quad (7.46)$$

**Step 6.** Lastly, we check that the dual vector fields  $\partial_{\theta_X^a}$  to the co-frame  $\{\theta_X, \hat{\eta}, \hat{\sigma}, \hat{\eta}, \hat{\sigma}\}$  on  $M$  and the dual vector fields  $\partial_{\theta_Y^a}$  to  $\{\theta_Y, \hat{\eta}, \hat{\sigma}, \hat{\eta}, \hat{\sigma}\}$  are related to the infinitesimal generators  $X_a$  and  $Y_a$  for the action of  $G$  on  $M_1$  and  $M_2$  by

$$\mathbf{q}_{M_1*}(X_a) = -\partial_{\theta_X^a} \quad \text{and} \quad \mathbf{q}_{M_2*}(Y_a) = \partial_{\theta_Y^a} \quad (7.47)$$

and satisfy

$$[\partial_{\theta_X^a}, \partial_{\theta_Y^a}] = 0. \quad (7.48)$$

For (7.47) it suffices to note, by (7.28), that  $X_a = \partial_{\theta_1^a}$  and  $Y_a = \partial_{\theta_2^a}$  and then to use equations (7.32) and (7.36) to calculate the Jacobians  $\mathbf{q}_{M_1*}$  and  $\mathbf{q}_{M_2*}$  in terms of the dual bases  $\{\partial_{\theta_1}, \partial_{\eta_1}, \partial_{\sigma_1}\}$  on  $M_1$  and  $\{\partial_{\theta_2}, \partial_{\eta_2}, \partial_{\sigma_2}\}$  on  $M_2$ .

To prove (7.48), let  $\mu_{1b}^c = (\lambda_1^{-1})_b^c$  and  $\mu_{2b}^c = (\lambda_2^{-1})_b^c$ . We then use (7.36) to calculate

$$\begin{aligned}\theta_X^a(\mathbf{q}_{G_{\text{diag}}}^*(\mu_{2b}^c \partial_{\theta_2^c})) &= (\mathbf{q}_{G_{\text{diag}}}^*(\theta_X^a))(\mu_{2b}^c \partial_{\theta_2^c}) = \delta_b^a \quad \text{and} \\ \theta_Y^a(\mathbf{q}_{G_{\text{diag}}}^*(\mu_{1b}^c \partial_{\theta_1^c})) &= (\mathbf{q}_{G_{\text{diag}}}^*(\theta_Y^a))(\mu_{1b}^c \partial_{\theta_1^c}) = -\delta_b^a.\end{aligned}$$

These equations, together with (7.32), lead to

$$\mathbf{q}_{G_{\text{diag}}}^*(\mu_{2b}^c \partial_{\theta_2^c}) = \partial_{\theta_X^b} \quad \text{and} \quad \mathbf{q}_{G_{\text{diag}}}^*(\mu_{1b}^c \partial_{\theta_1^c}) = -\partial_{\theta_Y^b} \quad (7.49)$$

from which it then follows, because the  $\mu_{1b}^c$  are functions on  $M_1$  and the  $\mu_{2b}^c$  are functions on  $M_2$ , that

$$[\partial_{\theta_X^a}, \partial_{\theta_Y^b}] = -\mathbf{q}_{G_{\text{diag}}}^*([\mu_{2a}^c \partial_{\theta_2^c}, \mu_{1b}^c \partial_{\theta_1^c}]) = 0. \quad (7.50)$$

This completes the proof that the co-frames  $\{\theta_\bullet, \hat{\eta}, \hat{\sigma}, \check{\eta}, \check{\sigma}\}$  are 4-adapted co-frames for the Darboux integrable system  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G$ .

Since the structure constants  $\mathbf{C}$  in (7.45) are the structure constants for the Lie algebra of  $G$ , this theorem immediately implies that *the Vessiot algebra for the Darboux integrable system  $(\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}$  is the Lie algebra of  $G$* . More generally, Theorems 6.1 and Lemma 6.7 combine to give the following result.

**Theorem 7.7.** *Under the hypothesis of Theorem 6.1, the Vessiot algebra of the Darboux integrable system  $(\mathcal{K}_1 + \mathcal{K}_2)/L$  is the Lie algebra of Lie group  $\tilde{L} = L/(A_1 \times A_2)$ .*

## 8 Quotient Representations for Darboux Compatible Integrable Extensions

In Theorem 6.1 we showed how Darboux integrable systems can be constructed using the group quotient of pairs of differential systems. It is a remarkable fact, established in [3], that the converse is true locally, that is, every Darboux integrable system can be locally realized as the quotient of a pair of differential systems with a common symmetry group. The precise formulation of this result is as follows.

**Theorem 8.1.** *Let  $(\mathcal{I}, M)$  be a Darboux integrable differential system with singular systems  $\hat{\mathcal{V}}$  and  $\check{\mathcal{V}}$ . Fix a point  $x_0$  in  $M$  and let*

- [i]  $M_1$  and  $M_2$  be the maximal integral manifolds of  $\hat{\mathcal{V}}^\infty$  and  $\check{\mathcal{V}}^\infty$  through  $x_0$ , and
- [ii]  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be the restrictions of  $\hat{\mathcal{V}}$  and  $\check{\mathcal{V}}$  to  $M_1$  and  $M_2$ .

*Then there are open sets  $U \subset M$ ,  $U_1 \subset M_1$ ,  $U_2 \subset M_2$ , each containing  $x_0$ , and a local action of a Lie group action  $G$  on  $U_1$  and  $U_2$ . This action satisfies the hypothesis of Theorem 6.1 and*

$$U = (U_1 \times U_2)/G_{\text{diag}} \quad \text{and} \quad \mathcal{I}|_U = (\mathcal{K}_1|_{U_1} + \mathcal{K}_2|_{U_2})/G_{\text{diag}}. \quad (8.1)$$

The Lie algebra of  $G$  coincides with the Vessiot Lie algebra of  $\mathcal{I}$  and the local actions of  $G$  on  $U_1$  and  $U_2$  are given by the restrictions of the Lie algebras of vectors fields  $\partial_{\theta_X}$  and  $\partial_{\theta_Y}$ , dual to the 4-adapted co-frames defined by Theorem 7.1. When these local actions can be extended to global actions of  $G$  on  $M_1$  and  $M_2$  and when  $M = (M_1 \times M_2)/G_{\text{diag}}$ , it then follows that

$$\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}. \quad (8.2)$$

In Section 8.1 we shall prove that the quotient representation (8.2) of  $\mathcal{I}$  is unique. We shall refer to (8.1) and (8.2) as *the local/global canonical quotient representation for a Darboux integrable differential system  $\mathcal{I}$* .

Suppose that  $\mathcal{I}$  be a Darboux integrable and that (8.2) holds. Let  $H$  be a subgroup of  $G$ , also satisfying the hypothesis of Theorem 6.1. Then, by Theorem A, we may construct the commutative diagram

$$\begin{array}{ccc} \mathcal{K}_1 + \mathcal{K}_2 & & \\ \mathbf{q}_{H_{\text{diag}}} \downarrow & \searrow \mathbf{q}_{G_{\text{diag}}} & \\ \mathcal{E} & \xrightarrow{\mathbf{p}} & \mathcal{I}, \end{array} \quad (8.3)$$

where  $\mathcal{E} = (\mathcal{K}_1 + \mathcal{K}_2)/H_{\text{diag}}$ . By Theorem A,  $\mathcal{E}$  is an integrable extension of  $\mathcal{I}$  and, by Theorem 6.4, the pair  $(\mathcal{E}, \mathcal{I})$  is Darboux compatible. We now state the following converse to Theorem 6.4 which precisely characterizes those integral extensions of Darboux integrable systems which are constructed by this group theoretic method.

**Theorem 8.2.** *Let  $(\mathcal{I}, M)$  be a Darboux integrable differential system with canonical quotient representation (8.2) and let  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  be an integrable extension. If  $(\mathcal{E}, \mathcal{I})$  is Darboux compatible, there there exists a subgroup  $H$  of  $G$  such that  $\mathcal{E} \cong_{\text{loc}} (\mathcal{K}_1 + \mathcal{K}_2)/H_{\text{diag}}$ .*

In Section 11, it will be shown that all integrable extensions of (level 2) Darboux integrable, Monge-Ampère equations in the plane are Darboux compatible and therefore can be constructed using Theorem 8.2. The proof of Theorem 8.2 is given in the next section.

## 8.1 The Proof of Theorem 8.2

In this section we tie together results on integrable extensions of Darboux integrable systems in Section 5 with the group reduction results of Section 6 to prove Theorem 8.2.

Let  $(\mathcal{E}, N)$  and  $(\mathcal{I}, M)$  be two Darboux integrable systems with quotient representations

$$\begin{aligned} (\mathcal{E}, N) &= ((\mathcal{L}_1 + \mathcal{L}_2)/H_{\text{diag}}, (Q_1 + Q_2)/H_{\text{diag}}) \quad \text{and} \\ (\mathcal{I}, M) &= ((\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}, (P_1 + P_2)/G_{\text{diag}}), \end{aligned}$$

where  $(\mathcal{L}_a, Q_a, H)$  and  $(\mathcal{K}_a, P_a, G)$  satisfy the hypothesis of Theorem 6.1. The singular systems for these differential systems are (see (6.7) and (6.8))

$$\begin{aligned} \hat{U} &= L_1^1 \oplus T^*Q_2, & \check{U} &= T^*Q_1 \oplus L_2^1, & \hat{Z} &= (L_1^1 \oplus T^*Q_2)/H_{\text{diag}}, & \check{Z} &= (T^*Q_1 \oplus L_2^1)/H_{\text{diag}}, \\ \hat{W} &= K_1^1 \oplus T^*P_2, & \check{W} &= T^*P_1 \oplus K_2^1, & \hat{V} &= (K_1^1 \oplus T^*P_2)/G_{\text{diag}}, & \check{V} &= (T^*P_1 \oplus K_2^1)/G_{\text{diag}} \end{aligned} \quad (8.4)$$

with derived systems (6.9). Theorem 6.1[iv] implies that, as integrable extensions,  $(\mathcal{L}_1 + \mathcal{L}_2, \mathcal{E})$  and  $(\mathcal{K}_1 + \mathcal{K}_2, \mathcal{I})$  are both Darboux compatible with respect to the singular systems (8.4).

Now suppose that there is a mapping  $\mathbf{p}: (\mathcal{E}, N) \rightarrow (\mathcal{I}, M)$  which defines  $(\mathcal{E}, N)$  as an integrable extension of  $(\mathcal{I}, M)$ . We therefore have the diagram

$$\begin{array}{ccc} (\mathcal{L}_1 + \mathcal{L}_2, Q_1 \times Q_2) & & (\mathcal{K}_1 + \mathcal{K}_2, P_1 \times P_2) \\ \mathbf{q}_{H_{\text{diag}}} \downarrow & & \downarrow \mathbf{q}_{G_{\text{diag}}} \\ (\mathcal{E}, N) & \xrightarrow{\mathbf{p}} & (\mathcal{I}, M). \end{array} \quad (8.5)$$

For a general integrable extension  $\mathbf{p}$ , little can be inferred about the relationship between  $(\mathcal{L}_a, Q_a, H)$  and  $(\mathcal{K}_a, P_a, G)$ . However, when  $(\mathcal{E}, \mathcal{I})$  are Darboux compatible with respect to the singular systems  $\{\hat{Z}, \check{Z}\}$  and  $\{\check{V}, \hat{V}\}$  (see Definition 5.2), we have the following important result. We assume here, for simplicity, that  $\mathcal{L}_a$  and  $\mathcal{K}_a$  are Pfaffian systems.

**Theorem 8.3.** *Let the diagram (8.5) be given. Suppose that the manifolds  $Q_a, P_a$  are connected, that the Lie group  $H$  is connected and that the actions of the groups  $H$  and  $G$  are free and regular on  $Q_1, Q_2$  and  $P_1, P_2$  respectively and transverse to  $\mathcal{L}_a$  and  $\mathcal{K}_a$ . Suppose that the differential systems  $(\mathcal{E}, \mathcal{I})$  are Darboux compatible with respect to the singular systems (8.4). Pick points  $(q_1, q_2) \in Q_1 \times Q_2$  and  $(p_1, p_2) \in P_1 \times P_2$  such that  $\mathbf{p} \circ \mathbf{q}_{H_{\text{diag}}}(q_1, q_2) = \mathbf{q}_{G_{\text{diag}}}(p_1, p_2)$  and set (see (6.4))*

$$\mathbf{q}_{Q_a} = \mathbf{q}_{H_{\text{diag}}} \circ \iota_{Q_a} : Q_a \rightarrow N \quad \text{and} \quad \mathbf{q}_{P_a} = \mathbf{q}_{G_{\text{diag}}} \circ \iota_{P_a} : P_a \rightarrow M. \quad (8.6)$$

Then there are globally defined local diffeomorphisms  $\psi_a : Q_a \rightarrow P_a$  and a Lie group homomorphism  $\phi : H \rightarrow G$  such that:

[i] the following two diagrams

$$\begin{array}{ccc} Q_1 & \xrightarrow{\psi_1} & P_1 \\ \mathbf{q}_{Q_1} \downarrow & & \downarrow \mathbf{q}_{P_1} \\ N & \xrightarrow{\mathbf{p}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} Q_2 & \xrightarrow{\psi_2} & P_2 \\ \mathbf{q}_{Q_2} \downarrow & & \downarrow \mathbf{q}_{P_2} \\ N & \xrightarrow{\mathbf{p}} & M \end{array} \quad (8.7)$$

are commutative;

[ii]  $\psi_a^*(\mathcal{K}_a) = \mathcal{L}_a$  so that the Pfaffian differential systems  $\mathcal{K}_a$  and  $\mathcal{L}_a$  are locally equivalent;

[iii] the maps  $\psi_a$  are  $H_{\text{diag}}$  equivariant in the sense that  $\psi_a(h \cdot q) = \phi(h) \cdot \psi_a(q)$  for all  $q \in Q_a$  and  $h \in H$ .

[iv] Assuming that  $H' = \phi(H) \subset G$  is a closed subgroup, [i], [ii] and [iii] together imply that

$$(\mathcal{E}, N) \cong_{\text{loc}} (\mathcal{K}_1 \times \mathcal{K}_2, P_1 \times P_2) / H'_{\text{diag}}. \quad (8.8)$$

*Proof.* By Theorem 6.1[v] the map  $\mathbf{q}_{Q_1} : Q_1 \rightarrow N$  is a maximal integral manifold for  $\hat{Z}^\infty$  through the point  $\mathbf{q}_{H_{\text{diag}}}(q_1, q_2)$ . By Remark 5.3,  $\mathbf{p}$  maps integral manifolds of  $\hat{Z}^\infty$  of maximal dimension to integral manifolds of  $\hat{V}^\infty$  of maximal dimension. Therefore  $\mathbf{p} \circ \mathbf{q}_{Q_1} : Q_1 \rightarrow N$  is an integral manifold of  $\hat{V}^\infty$  through  $\mathbf{p}(\mathbf{q}_{H_{\text{diag}}}(q_1, q_2))$  of maximum dimension  $\dim Q_1$ . Again, by Theorem 6.1[v], the map  $\mathbf{q}_{P_1} : P_1 \rightarrow M$  is a maximal integral manifold for  $\hat{V}^\infty$  through  $\mathbf{q}_{G_{\text{diag}}}(p_1, p_2)$  and therefore

$$\mathbf{p} \circ \mathbf{q}_{Q_1}(Q_1) \subset \mathbf{q}_{P_1}(P_1) \quad \text{and hence} \quad \dim Q_1 = \dim P_1. \quad (8.9)$$

The fact that  $\mathbf{q}_{P_1}$  is one-to-one (once more, Theorem 6.1[v]) then implies that  $\mathbf{p} \circ \mathbf{q}_{Q_1}$  factors through  $\mathbf{q}_{P_1}$ , that is, there is a unique map  $\psi_1 : Q_1 \rightarrow P_1$  such that the first diagram in (8.7) commutes. A basic result on the factorization of maps through integral manifolds ([37], page 47) states that  $\psi_1$  is smooth. Finally, since  $\mathbf{p} \circ \mathbf{q}_{Q_1}$  and  $\mathbf{q}_{P_1}$  both have injective differentials,  $\psi_{1*}$  is injective and hence  $\psi_1$  is a local diffeomorphism.

Equation (5.11) implies that

$$\mathbf{q}_{Q_1}^*(\hat{Z}) = \mathbf{q}_{Q_1}^*(\mathbf{p}^*(\hat{V}))$$

while equation (7.43) gives

$$L_1^1 = \mathbf{q}_{Q_1}^*(\hat{Z}) \quad \text{and} \quad K_1^1 = \mathbf{q}_{P_1}^*(\hat{V}). \quad (8.10)$$

In conjunction with the commutativity of (8.7) this leads to

$$\psi_1^*(K_1^1) = \psi_1^*(\mathbf{q}_{P_1}^*(\hat{V})) = \mathbf{q}_{Q_1}^*(\mathbf{p}^*(\hat{V})) = \mathbf{q}_{Q_1}^*(\hat{Z}) = L_1^1. \quad (8.11)$$

Similar arguments apply to the second diagram in (8.7) and the proof of parts [i] and [ii] are complete.

The proof of part [iii] is actually quite straightforward once all the appropriate notation is fixed. To this end, let  $X_i^Q, Y_i^Q$  be the infinitesimal generators for the action of  $G$  on  $Q_1$  and  $Q_2$  and let  $X_r^P, Y_r^P$  be the infinitesimal generators for the action of  $G$  on  $P_1$  and  $P_2$ . The structure equations are

$$[X_i^Q, X_j^Q] = C_{ij}^k X_k^Q, \quad [Y_i^Q, Y_j^Q] = C_{ij}^k Y_k^Q, \quad [X_r^P, X_s^P] = S_{rs}^t X_t^P, \quad [Y_r^P, Y_s^P] = S_{rs}^t Y_t^P.$$

The infinitesimal generators for the action of  $H_{\text{diag}}$  on  $Q_1 \times Q_2$  and  $G_{\text{diag}}$  on  $P_1 \times P_2$  are

$$V_i = X_i^Q + Y_i^Q \quad \text{and} \quad W_r = X_r^P + Y_r^P.$$

Since the actions of  $H_{\text{diag}}$  and  $G_{\text{diag}}$  are free and transverse to the differential systems  $\mathcal{L}_a$  and  $\mathcal{K}_a$ , we can apply Theorem 2.2 to obtain co-frames

$$\{\theta_{Q_a}, \eta_{Q_a}, \sigma_{Q_a}\} \quad \text{and} \quad \{\theta_{P_a}, \eta_{P_a}, \sigma_{P_a}\} \quad (8.12)$$



on  $Q_a$  and  $P_a$ . Note that  $\partial_{\theta_{Q_1}} = X^Q$ ,  $\partial_{\theta_{Q_2}} = Y^Q$ ,  $\partial_{\theta_{P_1}} = X^P$  and  $\partial_{\theta_{P_2}} = Y^P$ . Then from these co-frames we construct, as in Section 6.2 (Step 2 and Step 3), 4-adapted local co-frames

$$\{\boldsymbol{\theta}_{\bullet N}, \hat{\boldsymbol{\eta}}_N, \check{\boldsymbol{\eta}}_N, \hat{\boldsymbol{\sigma}}_N, \check{\boldsymbol{\sigma}}_N\} \quad \text{and} \quad \{\boldsymbol{\theta}_{\bullet M}, \hat{\boldsymbol{\eta}}_M, \check{\boldsymbol{\eta}}_M, \hat{\boldsymbol{\sigma}}_M, \check{\boldsymbol{\sigma}}_M\} \quad (8.13)$$

on the quotient manifolds  $N$  and  $M$ . In the present context equations (7.47) become

$$\mathbf{q}_{Q_1*}(X_i^Q) = -\partial_{\theta_{X,N}^i}, \quad \mathbf{q}_{Q_2*}(Y_i^Q) = \partial_{\theta_{Y,N}^i}, \quad \mathbf{q}_{P_1*}(X_r^P) = -\partial_{\theta_{X,M}^r}, \quad \mathbf{q}_{P_2*}(Y_r^P) = \partial_{\theta_{Y,M}^r}. \quad (8.14)$$

We now use (8.14) to prove that the maps  $\psi_a$  induce a common Lie algebra homomorphism  $\psi_{a,*} : \Gamma_H \rightarrow \Gamma_G$ , that is,

$$\psi_{1,*}(X_i^Q) = A_i^r X_r^P \quad \text{and} \quad \psi_{2,*}(Y_i^Q) = A_i^r Y_r^P, \quad (8.15)$$

where the  $A_i^r$  are constants. Since the integral extension  $\mathbf{p}$  is Darboux compatible  $\mathbf{p}^*(\mathcal{I}) \subset \mathcal{E}$ ,  $\mathbf{p}^*(\hat{V}) \subset \hat{Z}$  and  $\mathbf{p}^*(\check{V}) \subset \check{Z}$  and therefore, by virtue of the remarks made in Section 7.1 (see (7.6)), there are functions  $R_i^s$  and  $S_i^s$  on  $N$  such that

$$\mathbf{p}^*(\partial_{\theta_{X,N}^i}) = R_i^s \partial_{\theta_{X,M}^s} \quad \text{and} \quad \mathbf{p}^*(\partial_{\theta_{Y,N}^i}) = S_i^s \partial_{\theta_{Y,M}^s}. \quad (8.16)$$

If the vector fields  $\partial_{\theta_{X,N}^i}$ ,  $\partial_{\theta_{Y,N}^i}$  in (8.16) are evaluated at a point  $x \in N$ , then the functions  $R_i^s$  and  $S_i^s$  are evaluated at  $x$ . The combination of (8.14) and (8.16) thus yields

$$\mathbf{p}^*(\mathbf{q}_{Q_1*}(X_i^Q)) = -R_{0,i}^r \partial_{\theta_{X,M}^r} \quad \text{and} \quad \mathbf{p}^*(\mathbf{q}_{Q_2*}(Y_i^Q)) = S_{0,i}^r \partial_{\theta_{Y,M}^r}, \quad (8.17)$$

where  $R_{0,i}^r = R_i^r \circ \iota_{Q_1}$  and  $S_{0,i}^r = S_i^r \circ \iota_{Q_2}$ .

Since  $dR_i^r \in \hat{Z}^\infty$ , the functions  $R_i^r$  are constants on every integral manifold of  $\hat{Z}^\infty$ . But the map  $\iota_{Q_1} : Q_1 \rightarrow N$  is always an integral manifold of  $\hat{Z}^\infty$  (see Theorem 6.1[v]) and therefore the  $R_{0,i}^r$  are constants. Similarly,  $dS_i^r \in \check{Z}^\infty$ , the  $S_i^r$  are constants on every integral manifold of  $\check{Z}^\infty$  and therefore the  $S_{0,i}^r$  are constants. Consequently we can re-write (8.17) (using the last 2 equations from (8.14)) as

$$\mathbf{p}^*(\mathbf{q}_{Q_1*}(X_i^Q)) = \mathbf{q}_{P_1*}(R_{0,i}^r X_r^P) \quad \text{and} \quad \mathbf{p}^*(\mathbf{q}_{Q_2*}(Y_i^Q)) = \mathbf{q}_{P_2*}(S_{0,i}^r Y_r^P) \quad (8.18)$$

and therefore, by the commutative of the diagrams (8.7),

$$\psi_{1,*}(X_i^Q) = R_{0,i}^r X_r^P \quad \text{and} \quad \psi_{2,*}(Y_i^Q) = S_{0,i}^r Y_r^P. \quad (8.19)$$

To complete the proof of (8.15) it suffices to show that  $S_{0,i}^r = R_{0,i}^r$ . Equation (7.37) states

$$\boldsymbol{\theta}_{N,X}(q) = \boldsymbol{\theta}_{N,Y}(q), \quad \text{and} \quad \boldsymbol{\theta}_{M,X}(p) = \boldsymbol{\theta}_{M,Y}(p), \quad (8.20)$$

where  $q = \mathbf{q}_{H_{\text{diag}}}(q_1, q_2)$  and  $p = \mathbf{q}_{G_{\text{diag}}}(p_1, p_2)$ , and therefore  $\mathbf{p}^*\boldsymbol{\theta}_{M,X}(q) = \mathbf{p}^*\boldsymbol{\theta}_{M,Y}(q)$ . Using (8.20) in the definitions of  $\mathbf{R}$  and  $\mathbf{S}$  from equations (7.5) (with  $\phi = \mathbf{p}$ ) shows  $\mathbf{R}(q) = \mathbf{S}(q)$ , and hence  $S_{0,i}^r = R_{0,i}^r$ .

Equations (8.15) are valid on any local trivialization of  $Q_1$  and  $Q_2$  and therefore hold globally. We now deduce from Theorem 8.5 that there exists a homomorphism  $\phi: H \rightarrow G$  such that the  $\psi_a$  are  $\phi$  equivariant.

With the assumption that  $H' \subset G$  is closed, it follows (see Remark 3.3) that the action of  $H'_{\text{diag}}$  is regular on  $P_1 \times P_2$ . The equivariance of the maps  $\psi_a$  with respect to the actions of  $H$  and  $H'$  then induces a map  $\Psi: N \rightarrow (P_1 \times P_2)/H'$  such that the diagram

$$\begin{array}{ccc} Q_1 \times Q_2 & \xrightarrow{\psi_1 \times \psi_2} & P_1 \times P_2 \\ \mathbf{q}_H \downarrow & & \downarrow \mathbf{q}_{H'} \\ N & \xrightarrow{\Psi} & (P_1 \times P_2)/H' \end{array} \quad (8.21)$$

commutes. Since  $\psi_1 \times \psi_2$  is a local diffeomorphism, the function  $\Psi$  is as well. Since  $\psi_a^*(\mathcal{K}_a) = \mathcal{L}_a$ , a final application of Theorem 2.2 gives

$$\Psi^*((\mathcal{K}_1 + \mathcal{K}_2)/H') = (\mathcal{L}_1 + \mathcal{L}_2)/H = \mathcal{E}.$$

and part [iv] is established. ■

The next corollary establishes the uniqueness of the quotient representation of a Darboux integrable system.

**Corollary 8.4.** *Let  $(\mathcal{L}_a, Q_a, H)$  and  $(\mathcal{K}_a, P_a, G)$  satisfy the hypothesis of Theorem 6.1 and suppose, in addition that  $P_a, Q_a, H$  and  $G$  are all connected. If*

$$(Q_1 \times Q_2, \mathcal{L}_1 + \mathcal{L}_2)/H_{\text{diag}} \cong (P_1 \times P_2, \mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}, \quad (8.22)$$

*then the manifolds  $P_a$  and  $Q_a$  are diffeomorphic, the Lie groups  $H$  and  $G$  are isomorphic and  $\mathcal{L}_a \cong \mathcal{K}_a$ .*

*Proof.* We apply the Theorem 8.3 using  $\mathbf{p} = I_M$ , the identity map on  $M$ , to construct smooth maps  $\psi_a: Q_a \rightarrow P_a$ , and  $\tilde{\psi}_a: P_a \rightarrow Q_a$  which are inverses to each other. Similarly we have Lie group homomorphisms  $\phi: H \rightarrow G$  and  $\tilde{\phi}: G \rightarrow H$  which are inverses. ■

The following theorem was used in the proof of Theorem 8.3 .

**Theorem 8.5.** *Let  $H$  be a connected Lie group acting freely and regularly on  $N$  and let  $G$  be a Lie group acting freely and regularly on  $M$ . Suppose  $\Phi: N \rightarrow M$  satisfies*

$$\Phi_*: \Gamma_H \rightarrow \Gamma_G \quad (8.23)$$

*and is a monomorphism of the Lie algebras of infinitesimal generators. Then there exists a unique homomorphism  $\phi: H \rightarrow G$  such that  $\Phi$  is  $H$  equivariant, that is,  $\Phi(h \cdot p) = \phi(h) \cdot \Phi(p)$  for all  $p \in N$  and  $h \in H$ .*

## 9 Proof of Theorem B

We now combine Theorem A from the introduction with Theorem 6.1 to obtain the following general construction of Bäcklund transformations for a given Darboux integrable system  $\mathcal{I}$ .

**Theorem 9.1.** *Let  $(\mathcal{I}, M)$  be a Darboux integrable system with quotient representation  $(\mathcal{K}_1 + \mathcal{K}_2, M_1 \times M_2)/G_{\text{diag}}$ . Let  $L \subset G \times G$  be a subgroup, let  $H_{\text{diag}} = L \cap G_{\text{diag}}$  and suppose that the actions of  $L$  and  $H_{\text{diag}}$  are regular on  $M_1 \times M_2$ . Then the commutative diagram of differential systems*

$$\begin{array}{ccccc}
 & & (\mathcal{K}_1 + \mathcal{K}_2) & & \\
 & \swarrow \mathbf{q}_L & \downarrow \mathbf{q}_{H_{\text{diag}}} & \searrow \mathbf{q}_{G_{\text{diag}}} & \\
 & & (\mathcal{K}_1 + \mathcal{K}_2)/H_{\text{diag}} & & \\
 & \swarrow \mathbf{p}_1 & & \searrow \mathbf{p}_2 & \\
 (\mathcal{K}_1 + \mathcal{K}_2)/L & & & & (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}
 \end{array} \tag{9.1}$$

defines a Bäcklund transformation  $\mathcal{E} = (\mathcal{K}_1 + \mathcal{K}_2)/H_{\text{diag}}$  between  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}$  and the Darboux integrable system  $\mathcal{J} = (\mathcal{K}_1 + \mathcal{K}_2)/L$ .

We note that this theorem constructs a Bäcklund transformation with fiber dimensions  $\dim L - \dim H$  for  $\mathbf{p}_1$  and  $\dim G - \dim H$  for  $\mathbf{p}_2$ . With  $\hat{W}$  and  $\check{W}$  given by (6.7), Theorem 6.1[**ii**], gives the singular Pfaffian systems for  $\mathcal{I}$  and  $\mathcal{J}$  as

$$\hat{V} = \hat{W}/G_{\text{diag}}, \quad \check{V} = \check{W}/G_{\text{diag}}, \quad \hat{Z} = \hat{W}/L, \quad \check{Z} = \check{W}/L. \tag{9.2}$$

It is easy to check that when  $L \neq G_{\text{diag}}$ , the system  $\mathcal{J}$  has *strictly more* Darboux invariants than  $\mathcal{I}$ .

**Corollary 9.2.** *The integrable subsystems of the singular Pfaffian systems of  $\mathcal{I}$  and  $\mathcal{J}$  satisfy*

$$\text{rank}(\hat{Z}^\infty) \geq \text{rank}(\hat{V}^\infty) \quad \text{and} \quad \text{rank}(\check{Z}^\infty) \geq \text{rank}(\check{V}^\infty). \tag{9.3}$$

Moreover, if  $G$  is connected then we have equalities in (9.3) if and only if  $\mathcal{J} = \mathcal{I}$ .

*Proof.* Equations (6.10), when applied successively first to the diagonal subgroup  $G_{\text{diag}}$  and then to the group  $L$  gives

$$\begin{aligned}
 \text{rank}(\hat{V}^\infty) &= \dim M_2 - \dim G, & \text{rank}(\check{V}^\infty) &= \dim M_1 - \dim G & \text{and} \\
 \text{rank}(\hat{Z}^\infty) &= \dim M_2 - \dim L_2, & \text{rank}(\check{Z}^\infty) &= \dim M_1 - \dim L_1.
 \end{aligned}$$

The combination of these equations then yield

$$\text{rank}(\hat{Z}^\infty) = \text{rank}(\hat{V}^\infty) + \dim G - \dim L_1 \quad \text{and} \quad \text{rank}(\check{Z}^\infty) = \text{rank}(\check{V}^\infty) + \dim G - \dim L_2. \tag{9.4}$$

Since  $L_a = \rho_a(L) \subset G$  (see EqRefLaction), we have  $\dim L_a \leq \dim G$ . This proves (9.3). Equality holds in (9.3) only when  $\dim L_1 = \dim L_2 = \dim G$ . If  $G$  is connected this implies  $L_1 = L_2 = G$  and  $L$  is the diagonal subgroup.  $\blacksquare$

**Corollary 9.3.** *Let*

$$\begin{array}{ccc} & \mathcal{B} & \\ \mathbf{p}_1 \swarrow & & \searrow \mathbf{p}_2 \\ \mathcal{J} & & \mathcal{I} \end{array} \quad (9.5)$$

*define a Bäcklund transformation  $\mathcal{B}$  between differential systems  $\mathcal{I}$  and  $\mathcal{J}$ . Suppose  $\mathcal{I}$  is Darboux integrable and that the pair  $(\mathcal{B}, \mathcal{I})$  is Darboux compatible. If  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}$ , then there is subgroup  $H \subset G$  such that  $\mathcal{B} \cong_{\text{loc}} (\mathcal{K}_1 + \mathcal{K}_2)/H$  in which case the right half of the diagram (9.5) coincides (locally) with the right half of (1.2).*

*Proof.* This is just a restatement of Theorem 8.3.  $\blacksquare$

**Remark 9.4.** In the extreme case where  $L = G \times G$ , one has

$$(\mathcal{K}_1 + \mathcal{K}_2)/L = \mathcal{K}_1/G + \mathcal{K}_2/G,$$

$H_{\text{diag}} = G_{\text{diag}}$ , the projection map  $\mathbf{p}_2$  is the identity and the Vessot group for  $(\mathcal{K}_1 + \mathcal{K}_2)/L$  is the identity group. As an integrable extension, the integral manifolds of  $\mathcal{I} = (\mathcal{K}_1 + \mathcal{K}_2)/G_{\text{diag}}$  are obtained directly from integral manifolds of  $(\mathcal{K}_1 + \mathcal{K}_2)/L$  by solving ODE. *This simple remark effectively describes the entire classical integration method of Darboux integrable equations.*

## 10 Examples

In this section we shall use the group theoretical methods provided by Theorems A and C to construct Bäcklund transformations for a variety of Darboux integrable differential systems. In each example we shall begin with a differential system  $\mathcal{I}$  given as a direct sum  $\mathcal{K}_1 + \mathcal{K}_2$  on  $M_1 \times M_2$ . We define a group action  $H$  acting diagonally on  $M_1 \times M_2$  which is a symmetry group of  $\mathcal{I}$  and which acts transversally and, from this action, calculate the quotient differential system  $\mathcal{B} = \mathcal{I}/H$ . We then pick two more Lie symmetry groups  $G_1$  and  $G_2$  of  $\mathcal{I}$ , with  $H \subset G_1 \cap G_2$  and calculate the quotient differential systems  $\mathcal{I}_1 = \mathcal{I}/G_1$  and  $\mathcal{I}_2 = \mathcal{I}/G_2$ . The orbit projection maps  $\mathbf{p}_a : \mathcal{B} \rightarrow \mathcal{I}_a$  define the sought after Bäcklund transformation.

In Example 10.1, we show how this approach effortlessly gives the Bäcklund transformations between various Darboux integrable  $f$ -Gordon equations constructed in [14] and [38]. Examples 10.3 and 10.4 show that our method also gives Bäcklund transformations for non-Monge-Ampère equations and for PDE which are Darboux integrable at higher jet order. Bäcklund transformations are constructed for the  $A_2$  Toda lattice systems in Example 10.6. A Bäcklund transformation for an over-determined systems of PDE is given in Example 10.6.

**Example 10.1.** We take for  $\mathcal{I}$  the standard contact system  $\mathcal{K}_1 + \mathcal{K}_2$  on  $M = J^2(\mathbf{R}, \mathbf{R}) \times J^2(\mathbf{R}, \mathbf{R})$  with coordinates  $(x, v, v_1, v_2, y, w, w_1, w_2)$ . The total derivative vector fields defined by  $\mathcal{I}$  are

$$D_x = \partial_x + v_1 \partial_v + v_2 \partial_{v_1} \quad \text{and} \quad D_y = \partial_y + w_1 \partial_w + w_2 \partial_{w_1}. \quad (10.1)$$

We consider the infinitesimal group action  $\Gamma_H$  on  $M$  defined by the prolongation of the vector fields

$$X_1 = \partial_v - \partial_w \quad \text{and} \quad X_2 = v \partial_v + w \partial_w. \quad (10.2)$$

By definition, these are symmetries of  $\mathcal{I}$ . The quotient Pfaffian system  $\mathcal{B} = \mathcal{I}/\Gamma_H$  is a rank 2 Pfaffian system on a 6-dimensional manifold  $N$ . In terms of coordinates  $(x, y, V, V_x, W, W_y)$ , the projection map  $\mathbf{q}_H: M \rightarrow N$  is

$$\begin{aligned} \mathbf{q}_H(x, v, v_1, v_2, y, w, w_1, w_2) = \\ [x = x, y = y, V = \log \frac{v_1}{v+w}, V_x = D_x V, W = \log \frac{w_1}{v+w}, W_y = D_y W] \end{aligned}$$

and quotient Pfaffian system  $B$  is

$$B = \{ \beta^1 = dV - V_x dx + e^W dy, \quad \beta^2 = dW + e^V dx - W_y dy \}. \quad (10.3)$$

Here we take the open set where  $v + w > 0$ ,  $v_1 > 0$  and  $w_1 > 0$  for the domain of  $\mathbf{q}_H$ . The structure equations for  $B$  are

$$\begin{aligned} d\beta^1 &= -\hat{\pi} \wedge \hat{\omega}, \quad d\beta^2 = -\check{\pi} \wedge \check{\omega} \quad \text{mod } B \quad \text{where} \\ \hat{\omega} &= dx, \quad \hat{\pi} = dV_x - e^{V+W} dy, \quad \check{\omega} = dy, \quad \check{\pi} = dW_y - e^{V+W} dx. \end{aligned}$$

It is easily checked that  $B$  is a Darboux integrable hyperbolic system (of class  $s = 2$ ) with first integrals  $\{x, V_x + e^V\}$  and  $\{y, W_y + e^W\}$ . The associated PDE system is

$$V_y = -e^W, \quad W_x = -e^V. \quad (10.4)$$

We remark that in this simple example the total vector fields (10.1) commute with the infinitesimal generators (10.2) so the higher order differential invariants for the action of  $\Gamma_H$  are given by the total derivatives of the lowest order invariants  $V = \log \frac{v_1}{v+w}$  and  $W = \log \frac{w_1}{v+w}$ .

To construct some Bäcklund transformations using Theorem A we add, in turn, to the infinitesimal group action  $\Gamma_H$  the prolongations of the vectors

$$Z_1 = \partial_v + \partial_w, \quad Z_2 = y \partial_w, \quad Z_3 = x \partial_v - y \partial_w, \quad Z_4 = v^2 \partial_v - w^2 \partial_w.$$

Let  $\Gamma_{G_i} = \{\Gamma_H, Z_i\}$ . The actions  $\Gamma_{G_3}, \Gamma_{G_4}$  are diagonal actions on  $J^2(\mathbf{R}, \mathbf{R}) \times J^2(\mathbf{R}, \mathbf{R})$  but  $\Gamma_{G_1}$  and  $\Gamma_{G_2}$  are not. We calculate the reductions  $(\mathcal{I}_i, M_i)$  of  $\mathcal{I}$  by  $\Gamma_{G_i}$  and the orbit projection maps  $\mathbf{p}_i: N \rightarrow M_i$ . Each  $M_i$  has dimension 5 and the  $\mathcal{I}_i$  are all type  $s = 1$  hyperbolic systems

(that is, generated by a single 1-form and a pair of 2-forms). The result is the 4-fold collection of integrable extensions  $\mathbf{p}_i: \mathcal{B} \rightarrow \mathcal{I}_i$  which, in terms of the associated partial differential equations, is given explicitly by

$$\begin{array}{c}
 \boxed{\mathcal{K}_1 + \mathcal{K}_2} \\
 \downarrow \mathbf{q}\Gamma_H \\
 \boxed{V_y = -e^W, \quad W_x = -e^V} \\
 \swarrow \mathbf{p}_1 \quad \searrow \mathbf{p}_2 \quad \swarrow \mathbf{p}_3 \quad \searrow \mathbf{p}_4 \\
 \begin{array}{ccc}
 \boxed{u_{1,xy} = 0} & \boxed{u_{2,xy} = \frac{1}{(u_2 - x)} u_{2,x} u_{2,y}} & \boxed{u_{3,xy} = \frac{2u_3 - x - y}{(u_3 - x)(u_3 - y)} u_{3,x} u_{3,y}} & \boxed{u_{4,xy} = e^{u_4}}
 \end{array}
 \end{array} \tag{10.5}$$

where

$$u_1 = V - W, \quad u_2 = \frac{ye^W + xe^V - 1}{e^V}, \quad u_3 = \frac{ye^W + xe^V - 1}{e^W + e^V}, \quad u_4 = V + W + \log 2. \tag{10.6}$$

These equations are to be supplemented with their total derivatives. *Each of the projections  $\mathbf{p}_i$  defines  $\mathcal{B}$  as an integrable extension so that any two of the equations in (10.5) are Bäcklund equivalent via the common Pfaffian system  $\mathcal{B}$ .*

Note that if we eliminate the variables  $V$  and  $W$  (and their derivatives) from the equations defining  $\mathbf{p}_1$  and  $\mathbf{p}_4$  (see (10.6)) we arrive at the usual Bäcklund transformation

$$u_{1,x} - u_{4,x} = \sqrt{2} \exp \frac{u_1 + u_4}{2}, \quad u_{1,y} + u_{4,y} = -\sqrt{2} \exp \frac{-u_1 + u_4}{2}.$$

relating the wave equation to the Liouville equation. The Bäcklund transformation relating the second and third equation in (10.5) is

$$u_3(u_3 - x)u_{2,x} - u_2(u_2 - x)u_{3,x} = 0, \quad u_3(u_3 - y)u_{2,y} + yu_2u_{3,y} = 0.$$

The Monge-Ampère system  $\mathcal{I}_1$  for the wave equation is Darboux integrable on the 5-manifold  $M_1$  while the other systems  $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$  are all well-known examples of Monge-Ampère equations which become Darboux integrable after one prolongation. In this regard, the calculations leading to (10.5) could also be done starting with  $\mathcal{I}^{[1]}$ , the canonical differential system on  $J^3(\mathbf{R}, \mathbf{R}) \times J^3(\mathbf{R}, \mathbf{R})$ . All of the hypothesis of Theorem 6.1 are now satisfied. In particular, the infinitesimal actions for the projections of the prolongations  $\text{pr} \Gamma_{G_i}$  to each individual  $J^3(\mathbf{R}, \mathbf{R})$  are free and transverse to the contact systems. Consequently, the *first integrals* for the singular Pfaffian systems, or *Darboux invariants*, for each of quotient systems  $\mathcal{I}_i^{[1]} = \mathcal{I}^{[1]} / \text{pr} \Gamma_{G_i}$  can be calculated in terms of the *group invariants* using Theorem 6.1[iii]. For example,  $\Gamma_{G_4}$  gives the standard action of  $\mathfrak{sl}(2)$  on the line

with group invariants  $x$  and  $(3v_3 - 2v_2^2)/v_1^2$  which reduce under  $\mathbf{q}_{G_4}$  to the Darboux invariants  $x$  and  $u_{4,xx} - \frac{1}{2}u_{4,x}^2$  for the Liouville equation  $u_{4,xy} = \exp(u_4)$ .

The infinitesimal group actions  $\Gamma_{G_3}$  and  $\Gamma_{G_4}$  are both diagonal actions. The number of first integrals for each of the singular systems  $\{\hat{V}_a, \check{V}_a\}$ ,  $a = 3, 4$ , is therefore  $(\dim J^3(\mathbf{R}, \mathbf{R}) - \dim \Gamma_{G_a}) = 2$  (see Theorem 6.1, equation (6.10)). The Vessiot algebras of  $\mathcal{I}_3^{[1]}$  and  $\mathcal{I}_4^{[1]}$  both have dimension 3 and, by Theorem 7.7, are isomorphic to the abstract Lie algebras defined by  $\Gamma_{G_3}$  and  $\Gamma_{G_4}$ . By Theorem 6.1 [iv] we know that for  $a = 3, 4$  the integrable extensions  $\mathbf{p}_a^{[1]}: \mathcal{B}_a^{[1]} \rightarrow \mathcal{I}_a^{[1]}$  are Darboux compatible.

The actions  $\Gamma_{G_1}$  and  $\Gamma_{G_2}$  are not diagonal. The singular systems for the wave equation  $\mathcal{I}_1^{[1]}$  each have 3 first integrals while the singular systems for  $\mathcal{I}_2^{[1]}$  have 2 and 3 first integrals. The Vessiot algebras for  $\mathcal{I}_1^{[1]}$  and  $\mathcal{I}_2^{[1]}$  have dimension 1 and 2 and can be determined from Theorem 7.7. For example, the infinitesimal action  $\Gamma_{G_1} = \{X_1, X_2, Z_1\} = \{\partial_u, \partial_v, u\partial_u + v\partial_v\}$  used to construct the reduction  $\mathbf{q}_1: \mathcal{K}_1 + \mathcal{K}_2 \rightarrow \mathcal{I}_1$  can be reduced to a diagonal action using Lemma 6.7. One easily computes  $\Gamma_{A_1} = \{\partial_u\}$ ,  $\Gamma_{A_2} = \{\partial_v\}$ , and  $\Gamma_{\tilde{L}_{\text{diag}}} = \Gamma_{G_1}/(\Gamma_{A_1} + \Gamma_{A_2}) = \{u\partial_u + v\partial_v\}$  and therefore the quotient map  $\mathbf{q}_{\Gamma_{G_1}}$  factors as

$$\begin{array}{ccc} J^2 \times J^2 & \xrightarrow{\mathbf{q}_{\Gamma_{A_1}} \times \mathbf{q}_{\Gamma_{A_2}}} & J^1 \times J^1 \\ & \searrow \mathbf{q}_{\Gamma_{G_1}} & \downarrow \mathbf{q}_{\Gamma_{\tilde{L}_{\text{diag}}}} \\ & & \mathcal{I}_1. \end{array} \quad (10.7)$$

The Vessiot algebra of  $\mathcal{B}^{[1]}$  (which, by Theorem 7.5, is the same as the Vessiot algebra of  $\mathcal{B}$ ) is the 2-dimensional solvable Lie algebra and, by Theorem 7.3, is a sub-algebra of each of the Vessiot algebras for  $\mathcal{I}_3^{[1]}$  and  $\mathcal{I}_4^{[1]}$ . The induced homomorphism on the Vessiot algebras defined by  $\mathbf{p}_1$  is not injective.

Finally, we remark that Theorem 9.1 and Corollary 9.2 apply to the pairs  $(\Gamma_{G_1}, \Gamma_{G_3})$ ,  $(\Gamma_{G_1}, \Gamma_{G_4})$ ,  $(\Gamma_{G_2}, \Gamma_{G_4})$  but not to  $(\Gamma_{G_2}, \Gamma_{G_3})$  or  $(\Gamma_{G_3}, \Gamma_{G_4})$ .

**Example 10.2.** We now demonstrate remark 9.4 using Liouville's equation and  $\Gamma_{G_4}$ . Let  $\Gamma_L$  be the prolongation of the vector-fields

$$\partial_v, v\partial_v, v^2\partial_v, \partial_w, w\partial_w, w^2\partial_w \quad (10.8)$$

to  $J^3(\mathbf{R}, \mathbf{R})$ . As noted above, the diagonal quotient  $(\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_{G_4}$  where  $\mathcal{K}_1 + \mathcal{K}_2$  are the standard contact systems on  $J^3(\mathbf{R}, \mathbf{R}) \times J^3(\mathbf{R}, \mathbf{R})$  is the seven manifold representation of Liouville's equation.

The prolongation of the action of  $\Gamma_{L_1}$  on  $J^3(\mathbf{R}, \mathbf{R})$  satisfies the conditions of Theorem 2.2 which gives the generators

$$\mathcal{K}^1 = \langle \theta_1, \theta_2, \theta_3, d\hat{s} \wedge dx \rangle_{\text{alg}} \quad (10.9)$$

where  $\sigma^1 = d\hat{s}$ ,  $\sigma_2 = dx$  and  $\hat{s}$  is the Schwartzian,

$$\hat{s} = \frac{v_3}{v_1} - \frac{3v_2^2}{v_1^2}.$$

Let  $\check{s}$  be the Schwartzian on the second copy of  $J^3(\mathbf{R}, \mathbf{R})$ .

The quotient  $(\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_L = \mathcal{K}_1/\Gamma_{L_1} + \mathcal{K}_2/\Gamma_{L_2}$  may now be computed by noting that the only semi-basic form in (10.9) is  $d\hat{s} \wedge dx$ . The reduction is then

$$(\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_L = \langle d\hat{s} \wedge dx, d\check{s} \wedge dy \rangle_{\text{alg}}.$$

The integral manifolds of  $(\mathcal{K}_1 + \mathcal{K}_2)/L$  with independence condition  $dx \wedge dy$  are given by

$$\hat{s} = f(x), \quad \check{s} = g(y). \quad (10.10)$$

The projection map  $\mathbf{p} : (\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_{G_4} \rightarrow (\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_L$  can be written using the coordinates from Example 10.1 as

$$\mathbf{p}(x, y, u_4, u_{4,x}, u_{4,y}, u_{4,xx}, u_{4,yy}) \rightarrow \left( x, y, \hat{s} = u_{4,xx} - \frac{1}{2}u_{4,x}^2, \check{s} = u_{4,yy} - \frac{1}{2}u_{4,y}^2 \right). \quad (10.11)$$

Now according to Remark 9.4 given an integral manifold (10.10), integral manifolds of  $(\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_{G_4} = \mathcal{I}_2^{[1]}$  can be found by determining integrable manifolds to the completely integrable system of  $\mathcal{I}^{[1]}$  on  $\mathbf{p}^{-1}(x, y, f(x), g(y))$ . By equation (10.11)

$$\mathbf{p}^{-1}(x, y, f(x), g(y)) = \{x, y, u_4, u_{4,x}, u_{4,y}, u_{4,xx} - \frac{1}{2}u_{4,x}^2 = f(x), \quad u_{4,yy} - \frac{1}{2}u_{4,y}^2 = g(y)\}, \quad (10.12)$$

and the integration of  $\mathcal{I}^{[1]}$  subject to these Ricatti equations is the classical integration method for Darboux integrable equations.

**Example 10.3.** Our next example is based upon results found in the PhD thesis of Francesco Strazzullo [32]. We start with the canonical Pfaffian systems  $\mathcal{K}_1$  and  $\mathcal{K}_2$  for the Monge equations

$$\frac{du}{ds} = F\left(\frac{d^2v}{ds^2}\right) \quad \text{and} \quad \frac{dy}{dt} = G\left(\frac{d^2z}{dt^2}\right) \quad \text{where } F_{v''} \neq 0 \text{ and } G_{z''} \neq 0. \quad (10.13)$$

On  $\mathbf{R}^5[s, u, v, v_1, v_2]$  and  $\mathbf{R}^5[t, y, z, z_1, z_2]$  these are given by

$$\begin{aligned} \mathcal{K}_1 &= \langle dv - v_1 ds, dv_1 - v_2 ds, du - F(v_2) ds \rangle_{\text{diff}} \quad \text{and} \\ \mathcal{K}_2 &= \langle dz - z_1 dt, dz_1 - z_2 dt, dy - G(z_2) dt \rangle_{\text{diff}} \end{aligned}$$

and we take  $\mathcal{I} = \mathcal{K}_1 + \mathcal{K}_2$  on  $M = \mathbf{R}^{10}$ . The conditions  $F_{v''} \neq 0$  and  $G_{z''} \neq 0$  imply that the derived flags for  $\mathcal{K}_1$  and  $\mathcal{K}_2$  have ranks  $[3, 2, 0]$ .

We shall calculate reductions of  $\mathcal{I}$  using the infinitesimal symmetries

$$\begin{aligned} X_1 &= \partial_v, \quad X_2 = s\partial_v + \partial_{v_1}, \quad X_3 = \partial_u, \quad Y_1 = \partial_z, \quad Y_2 = t\partial_z + \partial_{z_1}, \quad Y_3 = \partial_y, \\ Z_1 &= X_1 - X_2, \quad Z_2 = X_2 + Y_2, \quad Z_3 = X_3 - Y_3. \end{aligned} \quad (10.14)$$

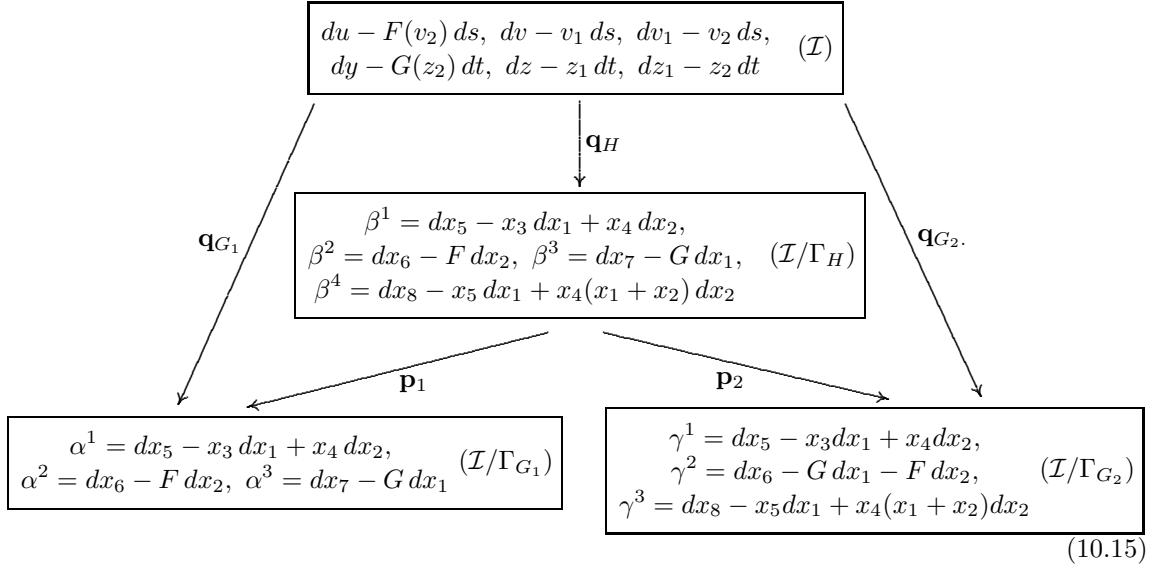
Let  $\Gamma_{G_1} = \{X_1, Y_1, Z_2\}$ ,  $\Gamma_{G_2} = \{Z_1, Z_2, Z_3\}$  and  $\Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \{Z_1, Z_2\}$ . These infinitesimal actions are all free and transverse to  $\mathcal{K}_1$  and  $\mathcal{K}_2$ . The actions  $\Gamma_H$  and  $\Gamma_{G_2}$  are diagonal actions while  $\Gamma_{G_1}$  is not.



Let  $N = \mathbf{R}^8[x_1, \dots, x_6, x_7, x_8]$ ,  $M_1 = \mathbf{R}^7[x_1, \dots, x_6, x_7]$  and  $M_2 = \mathbf{R}^7[x_1, \dots, x_6, x_8]$ . Then projection maps  $\mathbf{q}_H : M \rightarrow N$ ,  $\mathbf{q}_{G_1} : M \rightarrow M_1$  and  $\mathbf{q}_{G_2} : M \rightarrow M_2$  for the reduction by these 3 actions are

$$\begin{aligned}\mathbf{q}_H &= [x_1 = t, x_2 = s, x_3 = z_2, x_4 = v_2, x_5 = z_1 - v_1, x_6 = u, x_7 = y, x_8 = z + v - (s + t)v_1], \\ \mathbf{q}_{G_1} &= [x_1 = t, x_2 = s, x_3 = z_2, x_4 = v_2, x_5 = z_1 - v_1, x_6 = u, x_7 = y], \\ \mathbf{q}_{G_2} &= [x_1 = t, x_2 = s, x_3 = z_2, x_4 = v_2, x_5 = z_1 - v_1, x_6 = u + y, x_8 = z + v - (s + t)v_1]\end{aligned}$$

and the commutative diagram of Pfaffian systems (1.2) is easily computed to be



Here  $F = F(x_4)$  and  $G = G(x_3)$ . Theorems 6.1 and 9.1 apply directly to this diagram. All the differential systems here are therefore Darboux integrable, all the maps are integrable extensions, and the map  $\mathbf{p}_2$  defines  $\mathcal{I}/\Gamma_H$  as a Darboux compatible integrable extension of  $\mathcal{I}/\Gamma_{G_2}$ . The first integrals for each of the singular systems are easily calculated from the group invariants for each action. The Vessiot algebras are all abelian with  $\dim \mathbf{vess}(\mathcal{I}/\Gamma_{G_2}) = 3$ ,  $\dim \mathbf{vess}(\mathcal{I}/\Gamma_H) = 2$  and  $\dim \mathbf{vess}(\mathcal{I}/\Gamma_{G_1}) = 1$ .

These properties are all easily verified explicitly. For  $\mathcal{B} = \mathcal{I}/\Gamma_H$  the structure equations are

$$\begin{aligned}d\beta^1 &= G' \hat{\pi}^1 \wedge \hat{\pi}^2, & d\beta^2 &= F' \tilde{\pi}^1 \wedge \tilde{\pi}^2, & \text{mod } \{\beta^1, \beta^2\} \\ d\beta^3 &= \hat{\pi}^1 \wedge \hat{\pi}^2 - \tilde{\pi}^1 \wedge \tilde{\pi}^2, & d\beta^4 &= -(x_1 + x_2) \tilde{\pi}^1 \wedge \tilde{\pi}^2, & \text{mod } \{\beta^1, \beta^2\}\end{aligned}$$

where  $\hat{\pi}^1 = dx_1$ ,  $\hat{\pi}^2 = dx_3$ ,  $\tilde{\pi}^1 = dx_2$ ,  $\tilde{\pi}^2 = dx_4$ . This is a class  $s = 4$  hyperbolic Pfaffian system. The singular Pfaffian systems  $\{\beta^1, \beta^2, \hat{\pi}^1, \hat{\pi}^2\}$  and  $\{\beta^1, \beta^2, \tilde{\pi}^1, \tilde{\pi}^2\}$  have first integrals  $\{x_1, x_3, x_7\}$  and  $\{x_2, x_4, x_6\}$ . The derived system is

$$B' = \text{span}\left\{\beta^4 + \frac{x_1 + x_2}{F'}\beta^2, \beta^3 - \frac{1}{G'}\beta^1 + \frac{1}{F'}\beta^2\right\}.$$

These two forms in  $B'$  each define admissible sub-bundles for  $\mathcal{B}$  as integrable extensions of  $\mathcal{I}/\Gamma_{G_1}$  and  $\mathcal{I}/\Gamma_{G_2}$  respectively.

Just as in the previous example, the quotient map for the non-diagonal action factors through the reduction by the product action  $\{\partial_v, \partial_z\}$ . For the system  $\mathcal{I}_1 = \mathcal{I}/\Gamma_{G_1}$ , the structure equations are

$$d\alpha^1 = \hat{\pi}^1 \wedge \hat{\pi}^2 - \check{\pi}^1 \wedge \check{\pi}^2, \quad d\alpha^2 = G' \hat{\pi}^1 \wedge \hat{\pi}^2, \quad d\alpha^3 = F' \check{\pi}^1 \wedge \check{\pi}^2.$$

There are 3 first integrals for each singular Pfaffian system and hence, by a classical theorem of Lie, this implies that  $\mathcal{I}_1$  is (contact) equivalent to the wave equation. The change of variables

$$\begin{aligned} X' &= -x_6 + x_2 F, & Y' &= x_7 - x_1 G, & U' &= x_5 - x_1 x_3 + x_2 x_4, \\ P' &= \frac{1}{F'}, & Q' &= \frac{1}{G'}, & R' &= -\frac{F''}{x_2 F'^3}, & T' &= \frac{G''}{x_1 G'^3} \end{aligned} \quad (10.16)$$

transforms  $\mathcal{I}_1$  to the standard Pfaffian system for  $U'_{X'Y'} = 0$ .

To calculate the structure equations for  $\mathcal{I}_2 = \mathcal{I}/\Gamma_{G_2}$  we define a new co-frame by

$$\begin{aligned} \theta^1 &= (x_1 + x_2) G' \gamma^1 - (x_1 + x_2) \gamma^2 - (F' + G') \gamma^3, & \theta^2 &= F' \gamma^1 + \gamma^2, & \theta^3 &= -G' \gamma^1 + \gamma^2, \\ \hat{\pi}^1 &= dx_1 - \mu_2(x_1 + x_2) dx_3, & \hat{\pi}^2 &= dx_3, & \check{\pi}^1 &= dx_2 - \mu_1(x_1 + x_2) dx_4, & \check{\pi}^2 &= dx_4, \end{aligned} \quad (10.17)$$

where  $\mu_1 = \frac{F''}{F' + G'}$  and  $\mu_2 = \frac{G''}{F' + G'}$ . The resulting structure equations are

$$\begin{aligned} d\theta^1 &\equiv \theta^2 \wedge \hat{\pi}^1 + \theta^3 \wedge \check{\pi}^1, & \text{mod } \theta^1 \\ d\theta^2 &= (F' + G') \hat{\pi}^1 \wedge \hat{\pi}^2 - \mu_1(\theta^2 - \theta^3) \wedge \check{\pi}^2, \\ d\theta^3 &= (F' + G') \check{\pi}^1 \wedge \check{\pi}^2 + \mu_2(\theta^2 - \theta^3) \wedge \hat{\pi}^2. \end{aligned} \quad (10.18)$$

These structure equations show that  $\mathcal{I}_2$  defines a hyperbolic second order PDE. From the equations for  $d\theta^2$  and  $d\theta^3$  one can determine that the Monge-Ampère invariants for  $\mathcal{I}_2$  are proportional to  $F''$  and  $G''$ . Thus  $\mathcal{I}_2$  is of generic type (type (7, 7) in the terminology of [16]) when  $F'' \neq 0$  and  $G'' \neq 0$  and Monge-Ampère type when  $F'' = G'' = 0$ . The functions  $\{x_1, x_3\}$  and  $\{x_2, x_4\}$  are first integrals for the singular systems of  $\mathcal{I}_2$ .

In the special case of the Hilbert-Cartan equations

$$\frac{du}{ds} = \left(\frac{d^2 u}{ds^2}\right)^2 \quad \text{and} \quad \frac{dy}{dt} = \left(\frac{d^2 z}{dt^2}\right)^2 \quad (10.19)$$

we can use a recent theorem of D. The [34] to explicitly identify the PDE defined by  $\mathcal{I}_2$ . This theorem asserts that the equation  $3U_{XX}U_{YY}^3 + 1 = 0$  is uniquely characterized as the PDE in the plane with a 9-dimensional symmetry algebra with Levi decomposition  $\mathfrak{sl}(2) \ltimes \mathfrak{r}$  and such that the derived series for the radical  $\mathfrak{r}$  has dimensions  $[6, 5, 2]$ . Indeed, we find that the symmetry algebra

for  $\mathcal{I}_2$  is the 9-dimensional Lie algebra defined by

$$\begin{aligned}
X_1 &= x_3 \partial_{x_1} + x_4 \partial_{x_2} + \frac{1}{2}(x_3^2 - x_4^2) \partial_{x_5} + \frac{1}{3}(x_4^3 + x_3^3) \partial_{x_6} + (-\frac{1}{2}x_4^2 x_1 - \frac{1}{2}x_4^2 x_2 - \frac{1}{2}x_6 + x_5 x_3) \partial_{x_8}, \\
X_2 &= \frac{1}{2}x_1 \partial_{x_1} + \frac{1}{2}x_2 \partial_{x_2} - \frac{1}{2}x_3 \partial_{x_3} - \frac{1}{2}x_4 \partial_{x_4} - \frac{1}{2}x_6 \partial_{x_6} + \frac{1}{2}x_8 \partial_{x_8}, \\
X_3 &= -\frac{1}{2}x_1 \partial_{x_3} - \frac{1}{2}x_2 \partial_{x_4} + \frac{1}{4}(x_2^2 - x_1^2) \partial_{x_5} + (-x_1 x_5 + x_8) \partial_{x_6} + (-\frac{1}{12}x_1^3 + \frac{1}{4}x_2^2 x_1 + \frac{1}{6}x_2^3) \partial_{x_8}, \\
X_4 &= x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4} + 2x_5 \partial_{x_5} + 3x_6 \partial_{x_6} + 3x_8 \partial_{x_8}, \\
X_5 &= \partial_{x_5} + x_1 \partial_{x_8}, \quad X_6 = \frac{1}{2} \partial_{x_3} - \frac{1}{2} \partial_{x_4} + \frac{1}{2}(x_1 + x_2) \partial_{x_5} + x_5 \partial_{x_6} + \frac{1}{4}(x_1 + x_2)^2 \partial_{x_8}, \\
X_7 &= -\partial_{x_1} + \partial_{x_2}, \quad X_8 = \partial_{x_6}, \quad X_9 = \partial_{x_8}.
\end{aligned} \tag{10.20}$$

The radical of this Lie algebra is  $\{X_4, X_5, X_6, X_7, X_8, X_9\}$  with second derived algebra  $\{X_8, X_9\}$ . This algebraic data, together with the fact that  $\mathcal{I}_2$  is of generic type, suffices to characterize  $\mathcal{I}_2$  as the standard Pfaffian system for  $3U_{XX}U_{YY}^3 + 1 = 0$  ([34], section 7). By finding the diffeomorphism which maps the symmetries  $\{X_1, X_2, \dots, X_9\}$  to the symmetries of  $3U_{XX}U_{YY}^3 + 1 = 0$ , we are able to produce the following explicit contact equivalence:

$$\begin{aligned}
X &= -2\frac{x_3 + x_4}{x_1 + x_2}, \quad Y = x_5 - \frac{1}{2}(x_3 - x_4)(x_1 + x_2), \quad S = \frac{1}{2}(x_2 - x_1), \quad T = \frac{2}{x_1 + x_2}, \\
U &= 2\frac{x_3 + x_4}{x_1 + x_2}(-x_8 + x_1 x_5) - x_6 + \frac{1}{3}(2x_1 - x_2)x_4^2 - \frac{2}{3}(x_1 + x_2)x_3 x_4 - \frac{1}{3}(x_1 - 2x_2)x_3^2, \\
P &= x_8 - x_1 x_5 + \frac{1}{6}(x_1 + x_2)((2x_1 - x_2)x_3 - (x_1 - 2x_2)x_4), \quad Q = 2\frac{x_4 x_1 - x_3 x_2}{x_1 + x_2}.
\end{aligned} \tag{10.21}$$

The change of variables (10.16) and (10.21) allows us to rewrite the Bäcklund transformation (10.15) as

$$\begin{array}{ccc}
& \boxed{\begin{array}{l} dx_5 - x_3 dx_1 + x_4 dx_2 \\ dx_6 - x_4^2 dx_2, \quad dx_7 - x_3^2 dx_1 \\ dx_8 - x_5 dx_1 + x_4(x_1 + x_2) dx_2 \end{array}} & & \\
& \swarrow \mathbf{p}_1 & & \searrow \mathbf{p}_2 \\
\boxed{U_{XY} = 0} & & \boxed{3U_{XX}U_{YY}^3 + 1 = 0} .
\end{array} \tag{10.22}$$

It is interesting to remark that the identification of the quotient  $\mathcal{I}_2 = \mathcal{I}/G_2$  as the canonical Pfaffian system for  $3u_{xx}u_{yy}^3 + 1 = 0$  can actually be accomplished without explicitly calculating the reduction  $\mathcal{I}/G_2$  or its symmetry algebra. Indeed, we know that the symmetry algebra for  $\mathcal{I}$  is  $\mathfrak{g} = \mathfrak{g}_2 + \mathfrak{g}_2$ , two copies of the split real form of the exceptional Lie algebra  $\mathfrak{g}_2$ . For a basis for  $\mathfrak{g}$ , take

$$\{y_6, \dots, y_1, h_1, h_2, x_1, \dots, x_6, \tilde{y}_6, \dots, \tilde{y}_1, \tilde{h}_1, \tilde{h}_2, \tilde{x}_1, \dots, \tilde{x}_6\},$$

where the  $y_i$  are the negative roots, the  $h_i$  the Cartan sub-algebra and the  $x_i$  the positive roots. In terms of this basis, the sub-algebra defined by the infinitesimal generators  $\Gamma_{G_2}$  is

$$\mathfrak{h} = \{ x_2 - \tilde{x}_2, x_3 - \tilde{x}_3, x_6 + \tilde{x}_6 \}$$

and we find the normalizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  to be the 12-dimensional algebra

$$\text{nor}(\mathfrak{h}) = \{ y_5 - \tilde{y}_5, y_1 + \tilde{y}_1, h_1 + \tilde{h}_1, h_2 + \tilde{h}_2, x_2, x_3, x_4 - \tilde{x}_4, x_5 - \tilde{x}_5, x_6, \tilde{x}_2, \tilde{x}_3, \tilde{x}_6 \}.$$

From this we deduce that the symmetry algebra of  $\mathcal{I}/G_2$  is at least 9-dimensional. But because the equation is of generic type, the symmetry algebra is at most 9-dimensional and therefore the abstract Lie algebra for the symmetry algebra of  $\mathcal{I}_2$  is precisely  $\text{nor}(\mathfrak{h})/\mathfrak{h}$ .

**Example 10.4.** For each  $n = 0, 1, 2, \dots$  the equation

$$u_{xy} = \frac{2n \sqrt{pq}}{x + y} \quad (10.23)$$

is Darboux integrable at the  $n + 2$  jet level. We use this equation to show that the group theoretic approach to the construction of Bäcklund transformations is not limited to equations which are Darboux integrable on the 2-jet level.

Let  $\mathcal{G}_n$  be the standard differential system for (10.23). To construct Bäcklund transformations between  $\mathcal{G}_{n-1}$  and  $\mathcal{G}_n$ , we start with  $\mathcal{I} = \mathcal{H}_1^{n+1} + \mathcal{H}_2^{n+1}$ , where  $\mathcal{H}_1^{n+1}$  and  $\mathcal{H}_2^{n+1}$  are the standard rank  $n + 1$  differential systems for the  $n$ -th order Monge equations

$$\frac{du}{ds} = \left( \frac{d^n v}{ds^n} \right)^2 \quad \text{and} \quad \frac{dy}{dt} = \left( \frac{d^n z}{dt^n} \right)^2.$$

The vector fields

$$U = \partial_u, \quad V_i = s^i \partial_v, \quad Y = \partial_y, \quad Z_i = t^i \partial_z, \quad i = 0, 1, \dots, 2n - 1 \quad (10.24)$$

all lift to symmetries of  $\mathcal{I}$ . To apply Theorem A, we let

$$\Gamma_{G_1} = \{ U + V, V_0, Z_0, V_i + Z_i \}_{1 \leq i \leq 2n-2}, \quad \Gamma_{G_2} = \{ U + V, V_i + Z_i \}_{0 \leq i \leq 2n-1} \quad \text{and} \quad (10.25)$$

$$\Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \{ U + V, V_i + Z_i \}_{0 \leq i \leq 2n-2}. \quad (10.26)$$

Then we have the following commutative diagram of Pfaffian systems

$$\begin{array}{ccc} & \mathcal{H}_1^{n+1} + \mathcal{H}_2^{n+1} & \\ \mathbf{q}_{\Gamma_{G_1}} \swarrow & \downarrow \mathbf{q}_{\Gamma_H} & \searrow \mathbf{q}_{\Gamma_{G_2}} \\ & \mathcal{B}_n & \\ \swarrow \mathbf{p}_1 & & \searrow \mathbf{p}_2 \\ \boxed{U_{xy} = \frac{2(n-1) \sqrt{U_x U_y}}{x+y}} & & \boxed{V_{xy} = \frac{2n \sqrt{V_x V_y}}{x+y}} \end{array} \quad (10.27)$$

leading to the following theorem.

**Theorem 10.5.** *The differential equations*

$$U_{xy} = \frac{2(n-1)\sqrt{U_x U_y}}{x+y} \quad \text{and} \quad V_{xy} = \frac{2n\sqrt{V_x V_y}}{x+y} \quad (10.28)$$

*are related by the Bäcklund transformation*

$$(\sqrt{U_x} - \sqrt{V_x})^2 = \frac{(2n-1)(U-V)}{x+y} = (\sqrt{U_y} + \sqrt{V_y})^2. \quad (10.29)$$

Note that for  $n = 1$  this coincides with the Bäcklund transformation given in [38]. Detailed formulas for the various projection maps and quotients for (10.23) can be found in [2].

**Example 10.6.** As our final example we consider the  $A_2$  Toda lattice system

$$u_{xy} = 2e^u - e^v, \quad v_{xy} = -e^u + 2e^v. \quad (10.30)$$

This system is Darboux integrable and a Bäcklund transformation shall be constructed using Theorem A. By a simple linear change of variables, the  $A_2$  Toda lattice system can be rewritten as

$$u_{xy} = e^{2u-v}, \quad v_{xy} = e^{-u+2v}. \quad (10.31)$$

This latter formulation proves to be more amenable to our analysis.

The construction of the quotient representation for the  $A_2$  Toda lattice, based upon Theorem 8.1, has heretofore not appeared in the literature. We summarize the main steps here. The standard Pfaffian system  $\mathcal{I}$  for (10.31) is a rank 6 Pfaffian system on a 12-dimensional submanifold  $M$  of  $J^2(\mathbf{R}^2, \mathbf{R}^2)$ . To calculate the Vessiot group for (10.31) we must work at the prolongation order for which the system is Darboux integrable. However, the quotient representation and the Bäcklund transformation will be calculated as projections to  $M$ .

The first prolongation  $\mathcal{I}^{[1]}$  is a rank 10 Pfaffian system defined on a 16-dimensional submanifold  $M^{[1]}$  of  $J^3(\mathbf{R}^2, \mathbf{R}^2)$ . The associated singular Pfaffian system  $\hat{V}$  is generated by  $(I^{[1]})'$ , together with the three 1-forms  $dx$ ,

$$du_{xxx} - e^{2u-v}(2u_{xx} - v_{xx} + (2u_x - v_x)^2)dy, \quad dv_{xxx} - e^{2v-u}(2v_{xx} - u_{xx} + (-u_x + 2v_x)^2)dy.$$

The first integrals for  $\hat{V}$  are  $\hat{I}_1 = x$ ,  $\hat{I}_2 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2$ ,

$$\hat{I}_3 = u_{xxx} + u_x^2 v_x + (-2u_{xx} + v_{xx} - v_x^2)u_x, \quad \text{and} \quad \hat{I}_4 = v_{xxx} + u_x v_x^2 + (u_{xx} - 2v_{xx} - u_x^2)v_x.$$

By interchanging  $x$  with  $y$  in the above equations, we obtain the formulas for the other singular Pfaffian system  $\check{V}$  and its first integrals. We conclude that

$$\dim \hat{V} = \dim \check{V} = 13, \quad \dim \hat{V}^{(\infty)} = \dim \check{V}^{(\infty)} = 4, \quad \text{and} \quad \dim(\hat{V}^{(\infty)} \cap \check{V}) = \dim(\hat{V} \cap \check{V}^{(\infty)}) = 4$$

from which it then follows that the prolonged  $A_2$  Toda lattice system  $\mathcal{I}^{[1]}$  is Darboux integrable. By Theorem 8.1 the prolonged  $A_2$  Toda lattice system is the quotient of the product of two, rank 9

Pfaffian systems  $W_1$  and  $W_2$  on 12-dimensional manifolds by the diagonal action of an 8-dimension Lie group  $G$ . The calculation of the 4-adapted co-frame for  $\mathcal{I}^{[1]}$  shows that  $G$  is the special linear group  $A_2 = SL(3)$ . The Pfaffian system  $W_1$  is given by the restriction of  $\hat{V}$  to the zero set of the first integrals  $\hat{I}_a$  (see Theorem 8.1) and hence

$$W_1 = \{du - u_x dx, dv - v_x dx, du_x - u_{xx} dx, du_y - e^{2u-v} dx, dv_x - v_{xx} dx, dv_y - e^{2v-u} dx, du_{xx} - u_{xxx} dx, dv_{xx} - v_{xxx} dx, dv_{yy} - e^{2v-u}(2v_y - u_y) dx\}. \quad (10.32)$$

The dimensions of the derived systems of  $W_1$  and the dimensions of their Cauchy characteristics are

$$[\dim W'_1, \dots, \dim W_1^{(5)}] = [7, 5, 3, 1, 0] \text{ and } [\dim C(W'_1), \dots, \dim C(W_1^{(5)})] = [2, 4, 6, 9, 12].$$

These numerical invariants suffice to characterize  $W_1$  as the canonical contact system on the mixed jet space  $J^{4,5}(\mathbf{R}, \mathbf{R}^2)$ . At this point it remains to identify the precise form of the action of the Vessiot group  $SL(3)$  on  $J^{4,5}(\mathbf{R}, \mathbf{R}^2)$ . Once this is done, the joint differential invariants for the diagonal action of  $G = SL(3)$  on  $J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  then determines the quotient representation

$$\mathbf{q}_G : J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times J^{3,4}(\mathbf{R}, \mathbf{R}^2) \rightarrow \frac{J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times^{3,4}(\mathbf{R}, \mathbf{R}^2)}{SL(3)} \cong_{\text{loc}} M.$$

However, the standard coordinates for  $J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  lead to very long explicit formulas for  $\mathbf{q}_G$  which make our subsequent derivation of the Bäcklund transformation for the  $A_2$  Toda lattice quite complicated. A much better choice of coordinates for  $J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  is  $[x, u, u_1, u_2, u_3, v, v_1, v_2, v_3, v_4]$ , with

$$W_1 = \{du - u_1 dx, du_1 - u_2 dx, du_2 - u_3 dx, dv - v_1 du, dv_1 - v_2 du, dv_2 - v_3 du, dv_3 - v_4 du\}. \quad (10.33)$$

We leave it to the reader to check that (10.32) and (10.33) are equivalent Pfaffian systems. For our second copy of  $J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  we use coordinates  $[y, w, w_1, w_2, w_3, z, z_1, z_2, z_3, z_4]$ , with

$$W_2 = \{dw - w_1 dx, dw_1 - w_2 dx, dw_2 - w_3 dx, dz - z_1 dw, dz_1 - z_2 dw, dz_2 - z_3 dw, dz_3 - z_4 dw\}.$$

The integral manifolds for  $W_1$  and  $W_2$  are given by

$$u = f(x), u_1 = f'(x), v = g(f(x)), v_1 = g'(f(x)), w = h(y), z = k(h(y)), \dots \quad (10.34)$$

Note that the formulas for the total derivative vector fields and the prolongation formula for vector fields on  $J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  are slightly different in these coordinates.

The desired infinitesimal diagonal action of  $SL(3)$  on  $J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  is given by the prolongation of the vector fields

$$\begin{aligned} X_1 &= \partial_u + \partial_w, & X_2 &= \partial_v + \partial_z, & X_3 &= u \partial_u + w \partial_w, \\ X_4 &= v \partial_u + z \partial_w, & X_5 &= u \partial_v + w \partial_z, & X_6 &= v \partial_v + z \partial_z, \\ Z_1 &= u^2 \partial_u + uv \partial_v + w^2 \partial_w + wz \partial_z, & Z_2 &= uv \partial_u + v^2 \partial_v + wz \partial_w + z^2 \partial_z. \end{aligned}$$

The lowest order joint differential invariants for this action are

$$U^1 = \frac{1}{3} \log \frac{u_1^3 v_2 w_1^3 z_2^2}{((u-w)z_1 + z - v)^3} \quad \text{and} \quad U^2 = \frac{1}{3} \log \frac{-u_1^3 v_2^2 w_1^3 z_2}{((u-w)v_1 + z - v)^3}. \quad (10.35)$$

These equations, together with their total derivatives, give the quotient representation  $\mathbf{q}_G$  for the  $A_2$  Toda lattice.

The construction of the Bäcklund transformation for (10.31) now follows the construction of the Bäcklund transformation for Liouville's equation given in Example 10.1. Let

$$\begin{aligned} \Gamma_{G_1} &= \{ X_1, X_2, X_3, X_4, X_5, X_6, Z_3 = \partial_u - \partial_w, Z_4 = \partial_v - \partial_z \}, \\ \Gamma_{G_2} &= \Gamma_G = \{ X_1, X_2, X_3, X_4, X_5, X_6, Z_1, Z_2 \}, \\ \Gamma_H &= \Gamma_{G_1} \cap \Gamma_{G_2} = \{ X_1, X_2, X_3, X_4, X_5, X_6 \}. \end{aligned}$$

The lowest order joint differential invariants for the action of  $\Gamma_{G_1}$  on  $J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  are

$$V^1 = \log \frac{-2u_1 w_1 v_2 z_2}{(v_1 - z_1)^2} \quad \text{and} \quad V^2 = \log \frac{w_1^3 z_2}{u_1^3 v_2} \quad (10.36)$$

which imply that  $\mathcal{I}/G_1$  is the differential system for the decoupled Liouville-wave

$$V_{xy}^1 = e^{V^1}, \quad V_{xy}^2 = 0. \quad (10.37)$$

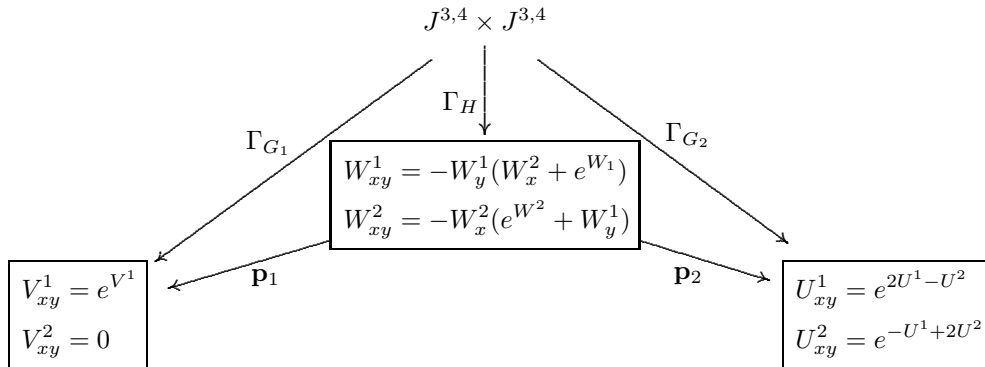
We next calculate the symmetry reduction of  $J^{3,4}(\mathbf{R}, \mathbf{R}^2) \times J^{3,4}(\mathbf{R}, \mathbf{R}^2)$  by the 6-dimensional Lie group with generators  $\Gamma_H$ . This will be a rank 8 Pfaffian system  $\mathcal{B}$  on a 14-dimensional manifold. For this action the lowest order joint differential invariants are the first order invariants

$$W^1 = \log \frac{u_1(v_1 - z_1)}{(w - u)z_1 + v - z} \quad \text{and} \quad W^2 = \log \frac{w_1(-z_1 + v_1)}{(w - u)v_1 + v - z}. \quad (10.38)$$

It then follows that the Pfaffian system  $\mathcal{B}$  is a partial prolongation of the standard Pfaffian system for the equations

$$W_{xy}^1 = -W_y^1(W_x^2 + e^{W_1}) \quad \text{and} \quad W_{xy}^2 = -W_x^2(e^{W^2} + W_y^1). \quad (10.39)$$

At this point we have constructed the Pfaffian systems for all of the equations in the following commutative diagram, as well as the quotient maps for the group actions  $\Gamma_H$ ,  $\Gamma_{G_1}$  and  $\Gamma_{G_2}$ .



We calculate the expressions for  $U^1, U^2, V^1, V^2$  in terms of  $W^1$  and  $W^2$  and their derivatives to be

$$\begin{aligned} U^1 &= \frac{2}{3}W^1 + \frac{1}{3}W^2 + \frac{1}{3}\log(-(W_y^1)^2W_x^2), & U^2 &= \frac{1}{3}W^1 + \frac{2}{3}W^2 + \frac{1}{3}\log(-W_y^1(W_x^2)^2), \\ V^1 &= \log(2W_y^1W_x^2), & V^2 &= 2W^2 - 2W^1 + \log\left(\frac{W_y^1}{W_x^2}\right). \end{aligned} \quad (10.40)$$

These formulas, together with their  $x$  and  $y$  derivatives to order 2, define the projection maps  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . It is a simple matter to check directly that  $\mathbf{p}_1$  and  $\mathbf{p}_2$  define integrable extensions. Finally, the elimination of the variables  $W^1$  and  $W^2$  from (10.40) lead to following first order PDE

$$\begin{aligned} V_x^1 - \frac{1}{3}V_x^2 - 2U_x^2 &= \sqrt{2}e^{(U^2 - \frac{1}{6}V^2 - U^1 + \frac{1}{2}V^1)}, & V_x^2 + 3U_x^1 &= -3\sqrt{2}e^{(-\frac{1}{2}V^1 + U^1 - \frac{1}{6}V^2)}, \\ V_y^1 + \frac{1}{3}V_y^2 - 2U_y^1 &= \sqrt{2}e^{(U^1 - U^2 + \frac{1}{2}V^1 + \frac{1}{6}V^2)}, & -V_y^2 + 3U_y^2 &= -3\sqrt{2}e^{(U^2 - \frac{1}{2}V^1 + \frac{1}{6}V^2)} \end{aligned}$$

for  $U^1, U^2, V^1, V^2$ , which give a Bäcklund transformation in the classical sense.

**Example 10.7.** In [3], several simple examples of Darboux integrable systems in 3 independent variables were given. Bäcklund transformations for all of these are easily constructed. For example, the system

$$u_{xz} = uu_x \quad u_{yz} = uu_y \quad (10.41)$$

is related to  $v_x = 0, v_y = 0$  by the Bäcklund transformation  $v_z = \exp(u)u_z$ .

## 11 An Application to Monge-Ampère systems in the plane

Let  $\mathcal{I}_2$  be a hyperbolic Monge-Ampère system on a 5-manifold  $M$  (see, for example, [14]) with singular Pfaffian systems  $\hat{V}_2$  and  $\check{V}_2$  satisfying<sup>3</sup>

$$\text{rank}(\hat{V}_2') = 2, \quad \text{rank}(\hat{V}_2^\infty) = 1, \quad \text{rank}(\check{V}_2') = 2, \quad \text{rank}(\check{V}_2^\infty) = 1. \quad (11.1)$$

Let  $\mathcal{I}_2^{[1]}$  be the prolongation of  $\mathcal{I}_2$  with respect to the independence condition  $\hat{V}_2^\infty \wedge \check{V}_2^\infty$  and suppose  $\mathcal{I}_2^{[1]}$  is Darboux integrable. The ranks of the singular Pfaffian systems  $\hat{V}_2^1, \check{V}_2^1$  for  $\mathcal{I}_2^{[1]}$  are  $\text{rank } \hat{V}_2^1 = 5$  and  $\text{rank } \check{V}_2^1 = 5$  while the assumption that  $\mathcal{I}_2^{[1]}$  is Darboux integrable implies that

$$\text{rank } \hat{V}_2^{1\infty} = 2 \quad \text{and} \quad \text{rank } \check{V}_2^{1\infty} = 2. \quad (11.2)$$

Equation (7.9) then shows that the dimension of the Vessiot algebra  $\mathbf{vess}(\mathcal{I}_2^{[1]})$  is 3.

Let  $\mathcal{I}_1$  be the Monge-Ampère form of the wave equation. Then  $\mathcal{I}_1$  is Darboux integrable (without prolongation) and the singular Pfaffian systems  $\hat{V}_1$  and  $\check{V}_1$  satisfy

$$\text{rank}(\hat{V}_1') = \text{rank}(\hat{V}_1^\infty) = 2 \quad \text{and} \quad \text{rank}(\check{V}_1') = \text{rank}(\check{V}_1^\infty) = 2. \quad (11.3)$$

---

<sup>3</sup>These rank conditions imply that  $\mathcal{I}$  is not Monge integrable after prolongation.



The Vessiot algebra for the wave equation has dimension 1.

The first theorem of this section shows that all the Bäcklund transformations explicitly constructed in [14] between such hyperbolic Monge-Ampère systems and the hyperbolic Monge-Ampère system for the wave equation arise as symmetry reductions of differential systems using Theorems 8.2 and 9.1.

**Theorem 11.1.** *Let  $(\mathcal{B}, N)$  be a (local) Bäcklund transformation, with 1-dimensional fibers, between a hyperbolic Monge-Ampère system  $(\mathcal{I}_2, M_2)$  and the Monge-Ampère system  $(\mathcal{I}_1, M_1)$  for the wave equation. Let  $\{\hat{V}_a, \check{V}_a\}$  and  $\{\hat{W}, \check{W}\}$  be the singular Pfaffian systems for  $\mathcal{I}_a$  and  $\mathcal{B}$  respectively. Suppose that  $\mathcal{I}_2^{[1]}$  is Darboux integrable and satisfies (11.1) and (11.2). Denote by  $\pi : (\mathcal{B}^{[1]}, N^{[1]}) \rightarrow (\mathcal{B}, N)$  and  $\pi_a : (\mathcal{I}_a^{[1]}, M_a^{[1]}) \rightarrow (\mathcal{I}_a, M_a)$  the prolongations of these systems.*

[i] *The prolongation  $\mathbf{p}_a^{[1]} : N^{[1]} \rightarrow M_a$  of the submersion  $\mathbf{p}_a : N \rightarrow M_a$  defines  $\mathcal{B}^{[1]}$  as an integrable extension of  $\mathcal{I}_a^{[1]}$ .*

[ii] *The differential system  $\mathcal{B}$  is a  $s = 2$  hyperbolic system. It is Darboux integrable,  $\hat{W}^\infty = \mathbf{p}_1^*(\hat{V}_1^\infty)$ ,  $\check{W}^\infty = \mathbf{p}_1^*(\check{V}_1^\infty)$  and the Vessiot algebra  $\mathbf{vess}(\mathcal{B})$  is 2-dimensional.*

[iii] *The integrable extension  $\mathbf{p}_2^{[1]} : \mathcal{B}^{[1]} \rightarrow \mathcal{I}_2^{[1]}$  is Darboux compatible.*

[iv] *There is a Lie algebra monomorphism from  $\mathbf{vess}(\mathcal{B})$  to the 3-dimensional algebra  $\mathbf{vess}(\mathcal{I}_2^{[1]})$ .*

[v] *The Bäcklund transformation  $\mathcal{B}^{[1]}$  can be constructed locally as a group quotient in accordance with Theorem 8.2.*

The mappings in part [i] of Theorem 11.1 define the commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{B}^{[1]} & & \\
 & \swarrow \mathbf{p}_1^{[1]} & \downarrow \pi & \searrow \mathbf{p}_2^{[1]} & \\
 \mathcal{I}_1^{[1]} & & \mathcal{B} & & \mathcal{I}_2^{[1]} \\
 \downarrow \pi_1 & \swarrow \mathbf{p}_1 & & \searrow \mathbf{p}_2 & \downarrow \pi_2 \\
 \mathcal{I}_1 & & & & \mathcal{I}_2
 \end{array} \tag{11.4}$$

The independence conditions on  $\mathcal{B}$  and  $\mathcal{I}_1$  are determined from the natural independence condition on  $\mathcal{I}_2$  as follows. Any non-zero 2-form  $\Omega_2$  in  $\hat{V}_2^\infty \wedge \check{V}_2^\infty$  fixes the independence condition for  $\mathcal{I}_2$ . The form  $\Omega = \mathbf{p}_2^*(\Omega_2)$  determines the independence condition for  $\mathcal{B}$ . This form can be taken to be  $\mathbf{p}_1$  basic so there is a unique form  $\Omega_1$  on  $M_1$  with  $\mathbf{p}_1^*(\Omega_1) = \Omega$  and this fixes the independence condition for  $\mathcal{I}_1$ . The prolongations in (11.4) are computed with respect to these independence conditions. It is a general fact that the fiber dimension for the prolongation of any class  $s$  hyperbolic system is 2. With no loss in generality we can label the singular Pfaffian systems so that

$$\mathbf{p}_a^*(\hat{V}_a) \subset \hat{W} \quad \text{and} \quad \mathbf{p}_a^*(\check{V}_a) \subset \check{W}. \tag{11.5}$$

We emphasize that Part [iii] of Theorem 11.1 is particular to the extension  $\mathbf{p}_2^{[1]} : \mathcal{B}^{[1]} \rightarrow \mathcal{I}_2^{[1]}$  and that Darboux compatibility does *not* hold for  $\mathcal{B}^{[1]}$  as an extension of the wave equation  $\mathcal{I}_1^{[1]}$ .

Our final result is at odds with Theorem 1 in [14].

**Theorem 11.2.** *Let  $\mathcal{I}_2$  be a hyperbolic Monge-Ampère system satisfying (11.1) and (11.2). If the Vessiot algebra  $\mathbf{vess}(\mathcal{I}_2^{[1]})$  is  $\mathfrak{so}(3)$ , then  $\mathcal{I}_2$  is not locally Bäcklund equivalent (with a Bäcklund transformation having a 1-dimensional fibers) to the wave equation. The Vessiot algebra for the equation*

$$u_{xy} = \frac{\sqrt{1-u_x^2}\sqrt{1-u_y^2}}{\sin u} \quad (11.6)$$

*is  $\mathfrak{so}(3)$  and therefore (11.6) is not Bäcklund equivalent (in the sense above) to the wave equation.*

Before considering the proofs of Theorems 11.1 and 11.2, a few general observations about integrable extensions of Monge-Ampère systems and their prolongations (which we shall apply to  $\mathcal{I}_1$  and  $\mathcal{I}_2$ ) are needed. The inequalities given below in (11.17) are especially important.

Let  $(\mathcal{K}, M)$  be a hyperbolic Monge-Ampère system whose singular Pfaffian systems  $\hat{V}$  and  $\check{V}$  satisfy (11.1) and (11.2) or (11.3). About each point  $p \in M$  there exists a neighborhood  $U$  of  $p$  and a local co-frame  $\{\theta, \hat{\tau}, \check{\tau}, \hat{\omega}, \check{\omega}\}$  on  $U$  such that

$$\mathcal{K}|_U = \langle \theta, \hat{\tau} \wedge \hat{\omega}, \check{\tau} \wedge \check{\omega} \rangle_{\text{alg}} \quad \text{and} \quad d\theta = \hat{\tau} \wedge \hat{\omega} + \check{\tau} \wedge \check{\omega} \quad \text{mod } \theta. \quad (11.7)$$

The singular Pfaffian systems of  $\mathcal{K}$  are given by

$$\hat{V} = \{\theta, \hat{\tau}, \hat{\omega}\} \text{ and } \check{V} = \{\theta, \check{\tau}, \check{\omega}\} \quad \text{and satisfy} \quad \hat{V}' = \{\hat{\tau}, \hat{\omega}\} \text{ and } \check{V}' = \{\check{\tau}, \check{\omega}\}. \quad (11.8)$$

We note that  $\hat{V}$  and  $\check{V}$  satisfy the constant rank conditions

$$\text{rank } \hat{V}'' = \text{rank } \hat{V}^\infty = c_1 \geq 1 \quad \text{rank } \check{V}'' = \text{rank } \check{V}^\infty = c_2 \geq 1. \quad (11.9)$$

(The specific values of these ranks are known for the Monge-Ampère systems  $\mathcal{I}_1$  and  $\mathcal{I}_2$  in Theorem 11.1 but are not the same.)

Let  $\mathbf{p}: (\mathcal{B}, N) \rightarrow (\mathcal{K}, M)$  be an integrable extension of  $\mathcal{K}$  with 1-dimensional fiber. On the open subset  $\mathbf{p}^{-1}(U) \subset N$ , let (by an abuse of notation)

$$\theta = \mathbf{p}^*(\theta), \quad \hat{\omega} = \mathbf{p}^*(\hat{\omega}), \quad \check{\omega} = \mathbf{p}^*(\check{\omega}), \quad \hat{\tau} = \mathbf{p}^*(\hat{\tau}), \quad \check{\tau} = \mathbf{p}^*(\check{\tau}).$$

For a local co-frame on  $N$  we choose an open set  $\tilde{U} \subset \mathbf{p}^{-1}(U)$  and extend the forms  $\{\theta, \hat{\tau}, \check{\tau}, \hat{\omega}, \check{\omega}\}$  to a co-frame on  $\tilde{U}$  with a 1-form  $\rho \in \mathcal{B}$  which is transverse to  $\mathbf{p}$ , that is,

$$\mathcal{B}|_{\tilde{U}} = \langle \rho, \theta, \hat{\tau} \wedge \hat{\omega}, \check{\tau} \wedge \check{\omega} \rangle_{\text{alg}}. \quad (11.10)$$

Since  $\mathcal{B}$  is an integrable extension of  $\mathcal{K}$

$$d\rho = a \hat{\tau} \wedge \hat{\omega} + b \check{\tau} \wedge \check{\omega} \quad \text{mod } \{\theta, \rho\}. \quad (11.11)$$

The singular Pfaffian systems for  $\mathcal{B}$  are then given, using equations (5.2), (11.7) and (11.11), by

$$\hat{W}|_{\tilde{U}} = \{\rho, \theta, \hat{\tau}, \hat{\omega}\} \quad \text{and} \quad \check{W}|_{\tilde{U}} = \{\rho, \theta, \check{\tau}, \check{\omega}\}. \quad (11.12)$$

The structure equations (11.7) and (11.11) show that  $\mathcal{B}$  is a  $s = 2$  hyperbolic system.

We now turn to the calculation of the derived flags for the singular systems  $\hat{W}$  and  $\check{W}$ . From (11.11) and (11.12) we find

$$\hat{W}'|_{\tilde{U}} = \{\rho - b\theta, \hat{\tau}, \hat{\omega}\} \quad \text{and} \quad \check{W}'|_{\tilde{U}} = \{\rho - a\theta, \check{\tau}, \check{\omega}\}. \quad (11.13)$$

On account of equations (11.8) and (11.13) and the fact that  $\rho$  is transverse to  $\mathbf{p}$ , it then follows that

$$\hat{W}'/\mathbf{p} = \hat{V}' \quad \text{and} \quad \check{W}'/\mathbf{p} = \check{V}'. \quad (11.14)$$

From the given structure equations alone it is not possible to calculate  $\hat{W}''$  and  $\check{W}''$  but we can derive bounds on the pointwise ranks of this spaces. Since reduction and derivation commute (see [4]), equations (11.2) and (11.14) imply that

$$\hat{W}''/\mathbf{p} = \hat{V}'' = \hat{V}^\infty \quad \text{and} \quad \check{W}''/\mathbf{p} = \check{V}'' = \check{V}^\infty$$

and therefore

$$\text{rank}(\hat{W}''_{\mathbf{p}, \mathbf{sb}}) = \text{rank} \hat{V}^\infty \quad \text{and} \quad \text{rank}(\check{W}''_{\mathbf{p}, \mathbf{sb}}) = \text{rank} \check{V}^\infty. \quad (11.15)$$

Since the fiber dimension of  $\mathbf{p}$  is 1, we have the simple inequalities

$$\text{rank} \hat{W}''_{\mathbf{p}, \mathbf{sb}} \leq \text{rank} \hat{W}'' \leq \text{rank} \hat{W}''_{\mathbf{p}, \mathbf{sb}} + 1 \quad (11.16)$$

which show that, for all  $p \in \tilde{U}$  and  $q = \mathbf{p}(p)$ ,

$$\text{rank} \hat{V}_q^\infty \leq \text{rank} \hat{W}_p'' \leq \text{rank} \hat{V}_q^\infty + 1 \quad \text{and, similarly,} \quad \text{rank} \check{V}_q^\infty \leq \text{rank} \check{W}_p'' \leq \text{rank} \check{V}_q^\infty + 1. \quad (11.17)$$

We shall use these inequalities for the proof of Theorem 11.1, part [ii].

For the special case under consideration here, the details for **IE[vii]** (Section 2.1) are easily given by identifying the local co-frames associated to the commutative diagram

$$\begin{array}{ccc} (\mathcal{B}^{[1]}, N^{[1]}) & \xrightarrow{\mathbf{p}^{[1]}} & (\mathcal{K}^{[1]}, M^{[1]}) \\ \pi_N \downarrow & & \downarrow \pi_M \\ (\mathcal{B}, N) & \xrightarrow{\mathbf{p}} & (\mathcal{K}, M) . \end{array} \quad (11.18)$$

To do so, let  $U^{[1]} = U \times \mathbf{R}^2[\hat{s}, \check{s}]$  be the prolongation space for  $\mathcal{K}$  over  $U \subset M$  with respect to the independence condition  $\hat{\omega} \wedge \check{\omega}$ . The 1-forms defining the co-frame on  $U$  pullback to  $U^{[1]}$  and allow us to define the local co-frame  $\{\theta, \theta^1 = \hat{\tau} + \hat{s}\hat{\omega}, \theta^2 = \check{\tau} + \check{s}\check{\omega}, \hat{\omega}, \check{\omega}, d\hat{s}, d\check{s}\}$  on  $U^{[1]}$ , with

$$\mathcal{K}^{[1]}|_{U^{[1]}} = \langle \theta, \theta^1, \theta^2 \rangle_{\text{diff}} . \quad (11.19)$$

The prolongation of  $\mathcal{B}$  is similarly constructed. Let  $\tilde{U}^{[1]} = \tilde{U} \times \mathbf{R}^2[\hat{t}, \check{t}]$  denote the subset of the prolongation space  $N^{[1]}$  with the independence condition  $\hat{\omega} \wedge \check{\omega}$ . We have

$$\mathcal{B}^{[1]}|_{\tilde{U}^{[1]}} = \langle \rho, \theta, \tilde{\theta}^1 = \hat{\tau} + \hat{t}\hat{\omega}, \tilde{\theta}^2 = \check{\tau} + \check{t}\check{\omega} \rangle_{\text{diff}}. \quad (11.20)$$

The prolongation  $\mathbf{p}^{[1]} : N^{[1]} \rightarrow M^{[1]}$  of  $\mathbf{p}$ , restricted to  $\tilde{U}^{[1]}$ , is

$$\mathbf{p}^{[1]}(p, \hat{t}, \check{t}) = (\mathbf{p}(p), \hat{s} = \hat{t}, \check{s} = \check{t}), \quad \text{for } p \in \tilde{U}^{[1]}. \quad (11.21)$$

The structure equation for  $\rho$  in (11.11) on  $\tilde{U}^{[1]}$  can be re-written using the forms in equation (11.20) as

$$d\rho = a\hat{\omega} \wedge \tilde{\theta}^1 + b\check{\omega} \wedge \tilde{\theta}^2 \quad \text{mod } \{\tilde{\theta}, \rho\}. \quad (11.22)$$

Equations (11.19), (11.20), and (11.22) show that  $\mathcal{B}^{[1]}$  is an integrable extension of  $\mathcal{K}^{[1]}$ .

With these preliminary computations in place, we can now prove Theorem 11.1.

*Proof of Theorem 11.1.* Part [i] of Theorem 11.1 follows directly from (11.18), with  $\mathcal{K} = \mathcal{I}_1$  and  $\mathcal{K} = \mathcal{I}_2$ .

To prove part [ii] we apply the inequalities (11.17) to each of the integrable extensions  $\mathbf{p}_a : (\mathcal{B}, N) \rightarrow (\mathcal{I}_a, M_a)$ . Firstly, the ranks for the singular Pfaffian systems  $\{\hat{V}_1, \check{V}_1\}$  for the wave equation are

$$\begin{aligned} \text{rank}(\hat{V}_1) &= 3, & \text{rank}(\hat{V}'_1) &= \text{rank}(\hat{V}''_1) = \text{rank}(\hat{V}_1^\infty) = 2, \\ \text{rank}(\check{V}_1) &= 3, & \text{rank}(\check{V}'_1) &= \text{rank}(\check{V}''_1) = \text{rank}(\check{V}_1^\infty) = 2 \end{aligned} \quad (11.23)$$

and hence, by (11.17),

$$2 \leq \text{rank}(\check{W}_p'') \leq 3 \quad \text{and} \quad 2 \leq \text{rank}(\hat{W}_p'') \leq 3. \quad (11.24)$$

Secondly, since the ranks for the derived systems for singular Pfaffian systems  $\{\hat{V}_2, \check{V}_2\}$  are given by (11.1), we also have, again by (11.17), that

$$1 \leq \text{rank}(\hat{W}_p'') \leq 2 \quad \text{and} \quad 1 \leq \text{rank}(\check{W}_p'') \leq 2. \quad (11.25)$$

From (11.24) and (11.25) we conclude

$$\text{rank}(\hat{W}''_p) = 2 \quad \text{and} \quad \text{rank}(\check{W}''_p) = 2. \quad (11.26)$$

Since  $\mathbf{p}_1^*(\hat{V}_1^\infty) \subset \hat{W}^\infty$  and  $\mathbf{p}_1^*(\check{V}_1^\infty) \subset \check{W}^\infty$ , we can then use (11.23) and (11.26) to conclude

$$\hat{W}^\infty = \mathbf{p}_1^*(\hat{V}_1^\infty) \quad \text{and} \quad \check{W}^\infty = \mathbf{p}_1^*(\check{V}_1^\infty). \quad (11.27)$$

Since  $\mathcal{B}$  is an integrable extension of  $\mathcal{I}_1$ , it is Darboux integrable (by Theorem 5.1) and moreover, by equations (7.9) and (11.27),  $\dim(\text{vers}(\mathcal{B})) = 2$ . Part [ii] of the theorem is proved.

We note that (11.27) also implies that the Darboux integrable systems  $\mathcal{B}$  and  $\mathcal{I}_1$  are *not* Darboux compatible.

To prove part [iii] we first use equations (7.19) and (11.12); (7.20) and (11.13); and (7.21) and (11.26) to verify that the ranks of the singular Pfaffian systems  $\hat{W}_1$  and  $\check{W}_1$  for  $\mathcal{B}^{[1]}$  are

$$\begin{aligned} \text{rank}(\hat{W}_1) &= 6, & \text{rank}(\hat{W}'_1) &= 5, & \text{rank}(\hat{W}_1^\infty) &= 3 & \text{ and} \\ \text{rank}(\check{W}_1) &= 6, & \text{rank}(\check{W}'_1) &= 5, & \text{rank}(\check{W}_1^\infty) &= 3. \end{aligned} \quad (11.28)$$

Now set  $\hat{J} = \hat{W}_1^\infty \cap \check{W}_1$  and  $\check{J} = \check{W}_1^\infty \cap \hat{W}_1$ . We shall verify that these are constant rank sub-bundles of  $T^*N^{[1]}$  which satisfy the conditions [ii] in the Definition 5.2 of Darboux compatibility. Since  $B^{[1]}$  is Darboux integrable, we have  $\hat{W}_1^\infty + \check{W}_1 = T^*N^{[1]}$  and  $\check{W}_1^\infty + \hat{W}_1 = T^*N^{[1]}$ , Equations (11.28) therefore imply that  $\text{rank } \hat{J} = 1$  and  $\text{rank } \check{J} = 1$ .

Accordingly, to prove that (see (5.9))

$$B^{[1]} = \hat{J} \oplus \mathbf{p}_2^{[1]*}(I_2^{[1]}), \quad \check{W}_1 = \hat{J} \oplus \mathbf{p}_2^{[1]*}(\check{V}_{21}) \quad \text{and} \quad \hat{W}_1^\infty = \hat{J} \oplus \mathbf{p}_2^{[1]*}(\check{V}_{21})$$

we need only verify that  $\hat{J}$  and  $\check{J}$  are not  $\mathbf{p}_2^{[1]}$  semi-basic at any point. In doing so we shall use detailed information about the left-hand side of the diagram (11.4) to deduce that the right-hand side is Darboux compatible.

Let  $\{\theta_1, \hat{\tau}_1, \hat{\omega}_1, \check{\tau}_1, \check{\omega}_1\}$  be an adapted co-frame for the wave equation, where  $\hat{V}_1^\infty|_U = \{\hat{\tau}_1, \hat{\omega}_1\}$ ,  $\check{V}_1^\infty = \{\check{\tau}_1, \check{\omega}_1\}$  and  $d\hat{\tau}_1 = 0, d\check{\tau}_1 = 0$ . The singular Pfaffian systems for  $\mathcal{B}$  are then (see (11.12))

$$\hat{W}|_{\tilde{U}} = \{\rho, \theta_1, \hat{\tau}_1, \hat{\omega}_1\} \quad \text{and} \quad \check{W}|_{\tilde{U}} = \{\rho, \theta_1, \check{\tau}_1, \check{\omega}_1\}. \quad (11.29)$$

and, by (11.27), the integrable subsystems for  $\hat{W}$  and  $\check{W}$  are

$$\hat{W}^\infty|_{\tilde{U}} = \{\hat{\tau}_1, \hat{\omega}_1\} \quad \text{and} \quad \check{W}^\infty|_{\tilde{U}} = \{\check{\tau}_1, \check{\omega}_1\}. \quad (11.30)$$

In other words, the co-frame  $\{\theta_1, \rho, \hat{\tau}_1, \hat{\omega}_1, \check{\tau}_1, \check{\omega}_1\}$  is a 0-adapted co-frame for the Darboux integrable system  $\mathcal{B}$  on  $N$ . Furthermore, (11.1) and (11.15), applied now to  $\mathbf{p}_2$ , imply that

$$\text{rank}(\hat{W}'')_{\mathbf{p}_2, \mathbf{s}\mathbf{b}} = 1 \quad \text{and} \quad \text{rank}(\check{W}'')_{\mathbf{p}_2, \mathbf{s}\mathbf{b}} = 1. \quad (11.31)$$

Thus  $\hat{\tau}_1$  and  $\check{\tau}_1$  are not  $\mathbf{p}_2$  semi-basic at any point in  $\tilde{U}$ .

Since the co-frame  $\{\theta, \rho, \hat{\tau}_1, \hat{\omega}_1, \check{\tau}_1, \check{\omega}_1\}$  is 0-adapted for  $\mathcal{B}$ , we can use the computations given in the proof of Theorem 7.5 (see equations (7.19) and (7.21)) to deduce that the singular Pfaffian systems for the prolongation  $\mathcal{B}^{[1]}$  satisfy

$$\hat{W}_1^\infty|_{\tilde{U}^{[1]}} = \{\hat{\tau}_1, \hat{\omega}_1, d\hat{s}\} \quad \text{and} \quad \check{W}_1|_{\tilde{U}^{[1]}} = \{\rho, \theta_1, \hat{\tau}_1 + \hat{s}\hat{\omega}_1, \hat{\tau}_1 + \check{s}\check{\omega}_1, \check{\omega}_1, d\check{s}\} \quad (11.32)$$

and therefore

$$\hat{J}|_{\tilde{U}^{[1]}} = (\hat{W}_1^\infty \cap \check{W}_1)|_{\tilde{U}^{[1]}} = \text{span}\{\hat{\tau}_1 + \hat{s}\hat{\omega}_1\}. \quad (11.33)$$

Because of our choice of independence conditions,  $\hat{\omega}_1$  is  $\mathbf{p}_2$  semi-basic at every point and so, because  $\hat{\tau}_1$  is not  $\mathbf{p}_2$  semi-basic at any point, the form  $\hat{\tau}_1 + \hat{s}\hat{\omega}_1$  is not  $\mathbf{p}_2^{[1]}$  semi-basic. Thus  $\hat{J}$  is transverse to  $\mathbf{p}_2^{[1]}$ , as required. A similar argument holds for  $\check{J}$  and so  $\mathcal{B}^{[1]}$ , as an integral extension of  $\mathcal{I}_2^{[1]}$ , is Darboux compatible.

Finally, parts [iv] and [v] of the theorem quickly follow from Corollary 7.4 and Theorem 8.2. ■

Finally, we give the proof of Theorem 11.2.

*Proof of Theorem 11.2.* The Lie algebra  $\mathfrak{so}(3)$  has no two dimensional subalgebra. Therefore by Theorem 11.1[iv], no integrable extension  $\mathcal{B}$  of  $\mathcal{I}$  with one-dimensional fibre exists.

To finish the proof of the theorem we now show equation (11.6) has  $\mathfrak{so}(3)$  as its Vessiot algebra. Let  $(x, y, u, u_x, u_y, u_{xx}, u_{yy})$  be the standard jet coordinates on the open set  $M^{[1]} \subset \mathbf{R}^7$ , where  $|u_x| < 1$  and  $|u_y| < 1$ . The classical Darboux invariants for equation (11.6) are

$$\hat{\xi} = \frac{u_{xx}}{\sqrt{1-u_x^2}} - \sqrt{1-u_x^2} \cot u, \quad \check{\xi} = \frac{u_{yy}}{\sqrt{1-u_y^2}} - \sqrt{1-u_y^2} \cot u. \quad (11.34)$$

In terms of coordinates  $(x, y, u, \alpha = \arcsin u_x, \beta = \arcsin u_y, \hat{\xi}, \check{\xi})$ , where  $-\pi/2 < \alpha, \beta < \pi/2$  the differential system  $\mathcal{I}^{[1]}$  is the Pfaffian system given by the 1-forms

$$\begin{aligned} \theta^1 &= \hat{\xi} dx - (\check{\xi} \cos u - \sin u \cos \beta) dy - d\alpha + \cos u d\beta \\ \theta^2 &= (\check{\xi} \sin u \sin \alpha + \sin \alpha \cos u \cos \beta - \cos \alpha \sin \beta) dy + \cos \alpha du - \sin u \sin \alpha d\beta \\ \theta^3 &= -dx - (\sin \beta \sin \alpha + \check{\xi} \cos \alpha \sin u + \cos \alpha \cos u \cos \beta) dy + \sin \alpha du + \cos \alpha \sin u d\beta. \end{aligned} \quad (11.35)$$

The co-frame  $(\theta^1, \theta^2, \theta^3, dx, d\hat{\xi}, dy, d\check{\xi})$  is 4-adapted and the structure equations are

$$\begin{aligned} d\theta^1 &= -\theta^2 \wedge \theta^3 - \theta^2 \wedge dx - dx \wedge d\hat{\xi} + \cos u dy \wedge d\check{\xi} \\ d\theta^2 &= \theta^1 \wedge \theta^3 + \theta^1 \wedge dx + \hat{\xi} \theta^3 \wedge dx - \sin \alpha \cos u dy \wedge d\check{\xi} \\ d\theta^3 &= -\theta^1 \wedge \theta^2 - \hat{\xi} \theta^2 \wedge dx + \cos \alpha \sin u dy \wedge d\check{\xi}. \end{aligned}$$

These equations imply that the Vessiot algebra for (11.6) is  $\mathfrak{so}(3)$ . ■

## A On the Prolongation of Integrable Extensions

In this section we prove statements **IE** [v] and **IE** [vii] from Section 2.1. The proof of **IE** [v] is simple. Equation (2.5) also shows that the co-dimension of  $E$  in  $T^*P$  and the co-dimension of  $I$  in  $T^*M$  are the same and so the maximal integrable manifolds of  $\mathcal{E}$  and  $\mathcal{I}$  have the same dimension. Furthermore since  $J$  satisfies the transversality condition (2.2) we have  $\text{ann}(E) \cap \ker(\mathbf{p}_*) = 0$  and so  $\mathbf{p}_*(\text{ann}(E)) \rightarrow \text{ann}(I)$  is an isomorphism at each point  $x_0 \in P$ .

We next turn to **IE**[vii] which we shall prove with a sequence of lemmas. Let  $\phi : N \rightarrow M$  be a submersion,  $\mathcal{E}$  be an integrable extension of  $\mathcal{I}$ , and let  $J \subset T^*N$  be an admissible sub-bundle for the extension. Note by transversality (2.2),  $\text{ann}(J)$  is horizontal, and  $TN = \ker(\phi_*) \oplus \text{ann}(J)$ . In particular, this implies  $\phi_* : \text{ann}(J) \rightarrow TM$  is a bundle map which is an isomorphism on the fibre.

**Lemma A.1.** *The map  $\phi_* : TN \rightarrow TM$  defines a bijection between  $k$ -dimensional integral elements of  $\mathcal{E}$  at  $x \in N$  and  $k$ -dimensional integral elements of  $\mathcal{I}$  at  $\phi(x)$ .*

*Proof.* Let  $x \in N$  and let  $E \subset T_x N$  be a  $k$ -dimensional integral element of  $\mathcal{E}$  at  $x$ . If  $\theta \in \mathcal{I}_x$  then  $\theta(\phi_*(E)) = \phi^*\theta(E) = 0$  because of the extension property  $\phi^*\theta \in \mathcal{E}$ . Therefore  $\phi_*(E)$  is an integral element of  $\mathcal{I}$  at  $\phi(x)$ . Now since  $E$  is an integral element of  $\mathcal{E}$  and  $\mathcal{J} \subset \mathcal{E}$ , we have  $E \subset \text{ann}(J_x)$ . By the transversality condition (2.2)  $E \cap \ker \phi_{x,*} = 0$ , and so  $\phi_*(E)$  and  $E$  have the same dimension. To show that  $\phi_*$  is onto let  $E' \subset T_{\phi(x)} M$  be a  $k$ -dimensional integral element of  $\mathcal{I}$  at  $\phi(x)$ . Let

$$E = \phi_*^{-1}(E') \cap \text{ann}(J) \subset T_x N, \quad (\text{A.1})$$

that is  $E$  is the subspace of  $\text{ann}(J)$  which maps by  $\phi_*$  to  $E'$ . Clearly by the last line in the first paragraph  $E$  and  $E'$  have the same dimension. We need to check that  $E$  is an integral element of  $\mathcal{E}$ . Since  $\mathcal{E}$  is generated algebraically by  $\mathcal{J}$  and  $\phi^*(\mathcal{I})$  we need to only check  $\phi^*\theta(E) = 0$  for all  $\theta \in \mathcal{I}_y$  and  $\rho(E) = 0$  for all  $\rho \in J_x$ . These are both trivially true and  $\phi_*(E) = E'$ . Therefore  $\phi_*$  defines a bijection on  $k$ -dimensional integral elements.  $\blacksquare$

Now suppose that  $\pi_N : G_k(\mathcal{E}) \rightarrow N$  and  $\pi_M : G_k(\mathcal{I}) \rightarrow M$  the space of  $k$  dimensional integral element are smooth bundles. As usual  $N^{[1]} = G_k(\mathcal{E})$  and  $M^{[1]} = G_k(\mathcal{I})$  denote the prolongation space for  $\mathcal{E}$  and  $\mathcal{I}$ . Lemma A.1 implies the following.

**Corollary A.2.** *There is a smooth submersion  $\phi^{[1]} : N^{[1]} \rightarrow M^{[1]}$  defined by*

$$\phi^{[1]}(E_x) = \phi_*(E_x) \quad (\text{A.2})$$

which gives rise to the commutative diagram

$$\begin{array}{ccc} N^{[1]} & \xrightarrow{\phi^{[1]}} & M^{[1]} \\ \pi_N \downarrow & & \downarrow \pi_M \\ N & \xrightarrow{\phi} & M \end{array} \quad (\text{A.3})$$

and  $\phi^{[1]}$  restricts to a diffeomorphism on the fibres.

Another way to state this is that  $N^{[1]} = \phi^*(M^{[1]})$ .

**Lemma A.3.** *The maps  $\phi_*^{[1]} : \ker(\pi_{N,*}) \rightarrow \ker(\pi_{M,*})$  and  $\pi_{N,*} : \ker(\phi_*^{[1]}) \rightarrow \ker(\phi_*)$  are isomorphisms, and*

$$\ker(\pi_{N,*}) \cap \ker(\phi_*^{[1]}) = 0. \quad (\text{A.4})$$

*Proof.* The first claim follows because  $\phi^{[1]}$  is a diffeomorphism on each fibre. Consider equation (A.4), and let  $Y \in \ker \pi_{N,*} \cap \ker \phi_*^{[1]}$ . This mean  $Y$  is  $\pi_N$  vertical and  $\phi_*^{[1]}Y = 0$ . But  $\phi^{[1]}$  is an isomorphism on vertical vectors, therefore  $Y = 0$ .

To prove the second claim, we begin with the fact that  $\ker \phi_* = \pi_{N,*}(\ker((\pi_M \circ \phi^{[1]})_*))$ . So if  $X \in \ker \phi_*$ , there exists  $Y \in \ker(\pi_M \circ \phi^{[1]})_*$  with  $\pi_{N,*}(Y) = X$ . By hypothesis  $\phi_*^{[1]}Y$  is  $\pi_M$  vertical,

and since by Corollary A.2  $\phi^{[1]}$  is a diffeomorphism on the fibre, there exists  $Z \in T_p M^{[1]}$  which is  $\pi_N$  vertical with  $\phi_*^{[1]}(Z) = Y$ . Then let  $Y' = Y - Z$ , and we have  $\pi_{N,*}(Y') = \pi_{N,*}(Y) = X$  because  $Z$  is vertical, and  $\phi_*^{[1]}(Y') = \phi_*^{[1]}(Y) - \phi_*^{[1]}(Z) = 0$ . This proves that  $\pi_{N,*}$  is onto. The fact that  $\pi_{N,*}$  is one-to-one follows easily from equation (A.4).  $\blacksquare$

**Corollary A.4.** *Let  $E_x \in T_x N$  be an integral element of  $\mathcal{E}$  at  $x \in N$  then*

$$\text{ann}(E_x) = \phi^* \text{ann}(\phi_*(E_x)) \oplus J_x. \quad (\text{A.5})$$

*Proof.* To prove equation (A.5) let  $E' = \phi_*(E_x)$  in equation (A.1) and take it's annihilator to get

$$\text{ann}(E_x) = \text{ann}(\phi_*^{-1}(\phi_*(E_x))) + J_x = \phi^*(\text{ann}(\phi_* E_x)) + J_x.$$

We then show these two spaces have trivial intersection. This follows from the transversality condition for  $J$ ,  $\phi^* TM \cap J = 0$ .  $\blacksquare$

Let  $\iota_N : G_k(\mathcal{E}) \rightarrow G_k(TN)$  and  $\iota_M : G_k(\mathcal{I}) \rightarrow G_k(TM)$  be the inclusion maps into the Grassmann bundles of  $k$ -planes, and let  $\mathcal{C}_N$  and  $\mathcal{C}_M$  be the canonical Pfaffian systems on  $G_k(TN)$  and  $G_k(TM)$  respectively. Then, by definition, the prolongation of  $\mathcal{E}$  and  $\mathcal{I}$  are the Pfaffian systems defined by  $\mathcal{E}^{[1]} = \iota_N^*(\mathcal{C}_N)$  and  $\mathcal{I}^{[1]} = \iota_M^*(\mathcal{C}_M)$ .

**Theorem A.5.** *With the submersion  $\phi^{[1]} : N^{[1]} \rightarrow M^{[1]}$  defined in equation (A.2), the EDS  $\mathcal{E}^{[1]}$  is an integral extension of  $\mathcal{I}^{[1]}$ . Moreover if  $J \subset T^*N$  is an admissible bundle for the extension  $\mathcal{E}$  of  $\mathcal{I}$ , then  $\pi_M^*(J)$  is an admissible bundle for the extension  $\mathcal{E}^{[1]}$  of  $\mathcal{I}^{[1]}$ .*

We prove this through two lemmas.

**Lemma A.6.** *The bundles  $E^{[1]}$  and  $I^{[1]}$  satisfy*

$$E^{[1]} = \pi_N^*(J) \oplus \phi^{[1],*}(I^{[1]}).$$

*Proof.* Let  $p = E_x \in N^{[1]}$  be an integral  $k$ -plane for  $\mathcal{E}$  at  $x \in N$ . By definition of the canonical Pfaffian system,

$$E_p^{[1]} = \pi_N^*(\text{ann}(E_x)) \quad (\text{A.6})$$

while by equation (A.2),

$$I_{\phi^{[1]}(p)}^{[1]} = \pi_M^*(\text{ann}(\phi_*(E_x))). \quad (\text{A.7})$$

Applying corollary A.4 and the commutative diagram (A.3) to equation (A.6), and using equation (A.7) gives

$$E_p^{[1]} = \pi_N^*(\phi^*(\text{ann}(\phi_* E_x)) \oplus J_x) = \phi^{[1],*} \pi_M^*(\text{ann}(\phi_* E_x)) \oplus \pi_N^*(J_x) = \phi^{[1],*} I_{\phi(p)}^{[1]} \oplus \pi_N^*(J_x),$$

which proves the lemma.  $\blacksquare$



**Lemma A.7.**  $\pi_M^* \mathcal{I} \subset \mathcal{I}^{[1]}$ .

*Proof of A.5.* We check the transversality condition. If  $Y \in \text{ann}(\pi_N^* J_x) \cap \ker \phi_*^{[1]}$  then by Corollary A.3  $\pi_{N,*} Y \in \ker \phi_*$ , and trivially  $\pi_{N,*} Y \in \text{ann}(J_x)$ . By the transversality of  $J$  we have  $\pi_{N,*} Y = 0$ . The fact that  $\pi_{N,*} : \ker \phi_*^{[1]} \rightarrow \phi_*$  is an isomorphism (Lemma A.3) implies  $Y = 0$ .

Lemma A.6 shows that Equation (2.2) holds at the bundle level. To show that equation (2.2) holds at the EDS level, we need to show if  $\sigma \in \mathcal{S}(\pi_N^* J)$  then  $d\sigma = 0 \mod E^{[1]}$ . This is a local question, so we may assume that we are working on an open set where

$$\sigma = A_a \pi_N^* \sigma^a \quad \sigma^a \in \mathcal{S}(J).$$

Then

$$d\sigma = A_a \pi_N^* d\sigma^a \mod \pi_N^* J.$$

However since  $J$  is admissible for the extension  $\mathcal{E}$  we have  $d\sigma_a = 0 \mod J + \phi^* \mathcal{I}$ . Since  $\pi_M^* \mathcal{I} \subset \mathcal{I}^{[1]}$ , by pulling  $d\sigma_a = 0 \mod J + \phi^* \mathcal{I}$  by  $\pi_N^*$ , and using the commutative diagram (A.3) we have

$$d\sigma = 0 \mod \pi_N^*(J) + \phi^{[1],*} \mathcal{I}^{[1]}.$$

This proves the theorem. ■

## B A Remark on Involutivity

We begin by recalling some terminology concerning involutive linear Pfaffian systems. Let  $\mathcal{I}$  be an involutive linear Pfaffian system on a manifold  $M$ . Then, about each point  $x \in M$ , there an open set  $U$ , a local basis  $\{\theta^a\}_{1 \leq a \leq m}$  for  $\mathcal{I}$  and a local co-frame  $\{\theta^a, \omega^i, \pi^\epsilon\}_{1 \leq i \leq n, 1 \leq \epsilon \leq p}$  for  $M$  such that

$$d\theta^a = A_{\epsilon i}^a \pi^\epsilon \wedge \omega^i \mod I. \quad (\text{B.1})$$

For the argument that follows we assume that the co-frame is chosen so that the tableaux  $A_{\epsilon i}^a$  has the form given by equation (90) of Chapter IV in [5] and we replace the generic  $\pi^\epsilon$  by the so-called principle components  $\bar{\pi}_i^a$ . Let  $s'_k$  be the last non-zero Cartan character. Then  $s'_1 + \dots + s'_k = p$  and we let

$$\theta_i^a = \bar{\pi}_i^a + p_{ij}^a \omega^j, \quad 1 \leq i \leq j \leq n \quad a \leq s'_k. \quad (\text{B.2})$$

The number of independent functions  $p_{ij}^a$  is  $t = s'_1 + 2s'_2 + \dots + ks'_k$  and these define the fibre coordinates on the set  $\pi : U^{[1]} \rightarrow U$ . A local basis for the prolongation of  $\mathcal{I}^{[1]}$  is then given by

$$\mathcal{I}^{[1]}|_{U^{[1]}} = \langle \theta^a, \theta_i^a \rangle_{\text{diff}} \quad (\text{B.3})$$

and a local co-frame on  $U^{[1]}$  is

$$T^*U^{[1]} = \text{span}\{\theta^a, \omega^i, \bar{\pi}_i^a, dp_{ij}^a\}. \quad (\text{B.4})$$

**Lemma B.1.** *Let  $\mathcal{I}^{[1]}$  be the prolongation of  $\mathcal{I}$  on  $M^{[1]}$ . Then the bundle of one-forms  $I^{[1]}$  is  $\pi$  semi-basic and  $\pi^* I \subset I^{[1]}$ . If  $\alpha \in \mathcal{I}^{[1]}$  is a one-form and  $d\alpha|_x$  is  $\pi$  semi-basic, then  $\alpha \in I_x$ .*

*Proof.* We need only demonstrate the last statement. Suppose  $\alpha$  is a one-form in  $\mathcal{I}^{[1]}$  satisfying the conditions in the statement of the lemma. About each point  $x \in M$  choose a local co-frame as in equation (B.4). There exists smooth functions  $T_a^i$  and  $S_a$  on  $U^{[1]}$  such that

$$\alpha|_{U^{[1]}} = \sum_{ai} T_a^i \theta_i^a + S_a \theta^a. \quad (\text{B.5})$$

In order for  $d\alpha$  to be  $\pi$  semi-basic at  $x$ , it is necessary and sufficient that

$$\partial_{p_{jk}^b} \lrcorner (dT_a^i \wedge \theta_i^a + T_a^i d\theta_i^a + dS_a \wedge \theta^a + S_a d\theta^a)|_x = 0. \quad (\text{B.6})$$

Since  $\theta_i^a$ ,  $\omega^i$  and  $\theta^a$  are  $\pi$  semi-basic we find that (B.6) simplifies to

$$\left( (\partial_{p_{jk}^b} \lrcorner dT_a^i) \theta_i^a + T_b^l \omega^k + (\partial_{p_{jk}^b} \lrcorner dS_a) \theta^a \right) |_x = 0. \quad (\text{B.7})$$

However  $\theta_i^a$ ,  $\omega^i$  and  $\theta^a$  are point-wise linearly independent and so this equation implies  $T_a^i(x) = 0$  and therefore  $\alpha|_x \in I_x$ . ■

**Corollary B.2.** *If  $\mathcal{I}$  is an involutive linear Pfaffian system, then  $\mathcal{I}^{[1]}/\pi = \mathcal{I}$ .*

**Remark B.3.** We now re-express Lemma B.1 in an alternative form using the co-frame  $\{\theta^a, \omega^i, \pi^\epsilon\}$  in equation (B.1). The prolongation is determined by

$$\tilde{\theta}^\epsilon = \pi^\epsilon + s^v S_{vi}^\epsilon \omega^i, \quad (\text{B.8})$$

where the smooth functions  $S_{vi}^\epsilon$  on  $U$  form a basis for the  $t$ -dimensional solution space of the linear system

$$A_{\epsilon i}^a S_j^\epsilon \omega^j \wedge \omega^i = 0.$$

The functions  $s^v$  define the local fibre coordinates on the prolongation space and  $\mathcal{I}^{[1]}|_{U^{[1]}} = \langle \theta^a, \tilde{\theta}^\epsilon \rangle$ .

We show that Lemma B.1 is equivalent to the statement that if  $k_\epsilon S_{vi}^\epsilon = 0$ , then  $k_\epsilon = 0$ . Suppose that  $k_\epsilon S_{vi}^\epsilon(x) = 0$  at some point  $x \in U$ . Then the exterior derivative of the form

$$\alpha = k_\epsilon \tilde{\theta}^\epsilon = k_\epsilon (\pi^\epsilon + s^v S_{vi}^\epsilon \omega^i)$$

is

$$d\alpha|_x = k_\epsilon (ds^v \wedge S_{vi}^\epsilon \omega^i + s^v dS_{vi}^\epsilon \wedge \omega^i + s^v S_{vi}^\epsilon d\omega^i)|_x = k_\epsilon s^v dS_{vi}^\epsilon \wedge \omega^i. \quad (\text{B.9})$$

Therefore  $d\alpha|_x$  is  $\pi$  semi-basic which by Lemma B.1 implies  $\alpha \in I_x$ . This is clearly possible only if only  $k_\epsilon = 0$ .

Conversely suppose that  $S_{vi}^\epsilon$  satisfy the above kernel condition. Let  $\alpha = T_\epsilon \tilde{\theta}^\epsilon + S_a \theta^a$  and suppose the  $d\alpha|_x$  is  $\pi$  semi-basic, that is,

$$\partial_{s^w} \lrcorner d\alpha_x = 0. \quad (\text{B.10})$$

On account of the fact the  $\pi^\epsilon$  and  $\theta^a$  are  $\pi$ -basic, equation (B.10) becomes

$$\left( (\partial_{s^w} \lrcorner dT_\epsilon) \tilde{\theta}^\epsilon + T_\epsilon S_{wi}^\epsilon \omega^i + (\partial_{s^w} \lrcorner dS_a) \theta^a \right) |_x = 0.$$

The forms  $\tilde{\theta}^\epsilon, \theta^a, \omega^i$  are independent, and so  $T_\epsilon S_{wi}^\epsilon = 0$ . By hypothesis  $T_\epsilon = 0$ , and so  $\alpha_x = (S_a \theta^a)|_x$ , which yields the conclusion of Lemma B.1.

**Lemma B.4.** *If  $\mathcal{I}$  is an involutive linear Pfaffian system, then  $I^{[1]'} = \pi^*(I)$ .*

*Proof.* Let  $\alpha \in I^{[1]}'$  and use the local co-frame (B.4) to write  $\alpha = T_a^i \theta_i^a + S_a \theta^a$ . Then

$$d\alpha = T_a^i dp_{il}^a \wedge \omega^l \mod I^{[1]}$$

and it then follows that  $T_a^i = 0$  and hence  $\alpha \in I$ . ■

## C On the definition of Darboux integrability

In this appendix we prove Theorem 4.3 The key step to proving Theorem 4.3 is to show that the decomposability of  $\mathcal{I}$ , together with conditions [i] of Definition 4.2, implies that  $\hat{V}^\infty \cap \check{V}^\infty$  is an integrable Pfaffian system. Then, since we are assuming that  $(\hat{V} \cap \check{V})^\infty = \{0\}$ , we deduce that  $\hat{V}^\infty \cap \check{V}^\infty \subset (\hat{V} \cap \check{V})^\infty = \{0\}$ .

To prove that  $\hat{V}^\infty \cap \check{V}^\infty$  is an integrable Pfaffian system we shall need a generalization of the 0-adapted co-frame defined in [3] (page 1917). This co-frame is defined locally in a neighborhood of any given point. First choose independent one-forms  $\tau = \{\tau^1, \tau^2, \dots, \tau^\ell\}$  such that

$$\hat{V}^\infty \cap \check{V}^\infty = \text{span}\{\tau\}.$$

Extend these by vector-valued (independent) 1-forms  $\hat{\eta}$  and  $\check{\eta}$  in such manner that

$$\hat{V}^\infty \cap \check{V} = \text{span}\{\tau, \hat{\eta}\} \quad \text{and} \quad \hat{V} \cap \check{V}^\infty = \text{span}\{\tau, \check{\eta}\}.$$

Then, just as in [3], these forms may in turn be extended (by conditions (4.6)) to a local co-frame of vector-valued 1-forms  $\{\theta, \hat{\sigma}, \check{\sigma}, \hat{\eta}, \check{\eta}, \tau\}$  on  $M$  such that

$$\begin{aligned} \hat{V} &= \text{span}\{\theta, \hat{\eta}, \check{\eta}, \tau, \hat{\sigma}\}, & \hat{V}^\infty &= \text{span}\{\hat{\eta}, \tau, \hat{\sigma}\}, \\ \check{V} &= \text{span}\{\theta, \hat{\eta}, \check{\eta}, \tau, \check{\sigma}\}, & \check{V}^\infty &= \text{span}\{\check{\eta}, \tau, \check{\sigma}\}. \end{aligned} \tag{C.1}$$

We will call such a coframe 0-adapted. The first step in the proof of Theorem 4.3 is given by the next lemma. In what follows we will use the convention that bold face Roman letters such as  $\mathbf{a}$ ,  $\mathbf{A}$ ,  $\boldsymbol{\alpha}, \dots$  denote array-valued functions and differential forms of the appropriate rank and dimensions.

**Lemma C.1.** *If  $\{\theta, \hat{\sigma}, \check{\sigma}, \hat{\eta}, \check{\eta}, \tau\}$  is a 0-adapted co-frame, then the forms  $\tau$  (which span  $\hat{V}^\infty \cap \check{V}^\infty$ ) satisfy the structure equations*

$$d\tau = \boldsymbol{\alpha} \wedge \tau + \mathbf{a}_1 \check{\eta} \wedge \hat{\eta} + \mathbf{a}_2 \check{\sigma} \wedge \hat{\eta} + \mathbf{a}_3 \check{\eta} \wedge \hat{\sigma}. \tag{C.2}$$

*Proof.* The conditions  $\tau \in \hat{V}^\infty$  and  $\tau \in \check{V}^\infty$  imply, by the Frobenius condition for integrability and equations (C.1), that there exists 1-forms  $\alpha, \beta, \gamma, \mu, \nu, \xi$  such that

$$d\tau = \alpha \wedge \tau + \beta \wedge \hat{\eta} + \gamma \wedge \hat{\sigma} \quad \text{and} \quad d\tau = \mu \wedge \tau + \nu \wedge \check{\eta} + \xi \wedge \check{\sigma}. \quad (\text{C.3})$$

Now the 1-forms  $\tau \in \mathcal{I}^1 = \text{span}\{\tilde{\theta}^e\}$  and so, by decomposability condition (4.5), there are no  $\hat{\sigma} \wedge \check{\sigma}$  terms in either of these structure equations. Hence, after adsorbing the  $\tau, \hat{\eta}$  terms in  $\gamma$  and the  $\tau, \check{\eta}$  terms in  $\xi$  into the other coefficients, we may re-write equations (C.3) in expanded form as

$$\begin{aligned} d\tau &= \alpha \wedge \tau + \beta \wedge \hat{\eta} + (c_1\theta + c_2\check{\eta} + c_3\hat{\sigma}) \wedge \hat{\sigma} \quad \text{and} \\ d\tau &= \mu \wedge \tau + \nu \wedge \check{\eta} + (C_1\theta + C_2\hat{\eta} + C_3\check{\sigma}) \wedge \check{\sigma}, \end{aligned} \quad (\text{C.4})$$

where the  $c_i$  and  $C_i$  are 3 dimensional arrays of locally defined smooth functions. For the right-hand sides of these equations to be equal we find immediately that  $c_1 = c_3 = C_1 = C_3 = 0$ . Upon adsorbing the  $\tau$  terms in  $\beta$  and  $\nu$  into  $\alpha$  and  $\mu$ , we can re-write equations (C.4) in expanded form as

$$\begin{aligned} d\tau &= \alpha \wedge \tau + (b_1\theta + b_2\hat{\eta} + b_3\check{\eta} + b_4\check{\sigma}) \wedge \hat{\eta} + c_2\check{\eta} \wedge \hat{\sigma}, \quad \text{and} \\ d\tau &= \mu \wedge \tau + (B_1\theta + B_2\hat{\eta} + B_3\check{\eta} + B_4\hat{\sigma}) \wedge \check{\eta} + C_2\hat{\eta} \wedge \check{\sigma}. \end{aligned} \quad (\text{C.5})$$

For the right-hand sides of these equations to be equal we further find that  $b_1 = b_2 = B_1 = B_3 = 0$  in which case equation (C.5) reduces to (C.2), as required.  $\blacksquare$

To complete the proof of Theorem 4.3 we must show that the coefficients  $a_1, a_2$  and  $a_3$  in (C.2) vanish and, to this end, we shall need a refined co-frame, generalizing the 1-adapted co-frame of [3]. Define

$$\text{Inv}(\hat{V}^\infty) = \{f \in C^\infty(M) \mid df \in \mathcal{S}(\hat{V}^\infty)\} \quad \text{and} \quad \text{Inv}(\check{V}^\infty) = \{f \in C^\infty(M) \mid df \in \mathcal{S}(\check{V}^\infty)\}.$$

**Lemma C.2.** *If  $f$  is a locally defined, real-valued function on  $M$  such that*

$$df \in \mathcal{S}(\hat{V}) = \text{span}\{\theta, \hat{\eta}, \check{\eta}, \hat{\sigma}, \tau\}$$

*then  $df \in \text{span}\{\hat{\eta}, \hat{\sigma}, \tau\}$ . Likewise, if  $df \in \mathcal{S}(\check{V}) = \text{span}\{\theta, \hat{\eta}, \check{\eta}, \check{\sigma}, \tau\}$  then  $df \in \text{span}\{\check{\eta}, \check{\sigma}, \tau\}$ .*

*Proof.* If  $df \in \mathcal{S}(\hat{V})$ , then, by the definition of the infinite derived Pfaffian system,  $df \in \mathcal{S}(\hat{V}^\infty)$ , and the lemma follows from equation (C.1).  $\blacksquare$

The next lemma gives a preferred 0-adapted co-frame.

**Lemma C.3.** *Let  $\{\theta, \hat{\sigma}, \check{\sigma}, \hat{\eta}, \check{\eta}, \tau\}$  be a 0-adapted co-frame. There exists vector-valued functions  $I_1, I_2, I_3 \in \text{Inv}(\hat{V}^\infty)$ , and  $J_1, J_2, J_3 \in \text{Inv}(\check{V}^\infty)$  and matrix-valued functions  $E$  and  $F$  such that the forms*

$$\hat{\sigma}_0 = dI_3, \quad \hat{\eta}_0 = dI_2 + E\hat{\sigma}_0, \quad \check{\sigma}_0 = dJ_3, \quad \check{\eta}_0 = dJ_2 + F\check{\sigma}_0 \quad (\text{C.6})$$

may be used to define a 0-adapted co-frame  $\{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}_0, \check{\boldsymbol{\eta}}_0, \hat{\boldsymbol{\sigma}}_0, \check{\boldsymbol{\sigma}}_0, \boldsymbol{\tau}\}$ . Moreover, the 1-forms  $\boldsymbol{\tau}$  can be expressed as

$$\boldsymbol{\tau} = \mathbf{R}(d\mathbf{I}_1 + \mathbf{S}\hat{\boldsymbol{\eta}}_0 + \mathbf{T}\hat{\boldsymbol{\sigma}}_0) = \mathbf{A}(d\mathbf{J}_1 + \mathbf{B}\hat{\boldsymbol{\eta}}_0 + \mathbf{C}\check{\boldsymbol{\sigma}}_0), \quad (\text{C.7})$$

where  $\mathbf{R}, \mathbf{A}, \mathbf{S}, \dots$  are smooth matrix-valued functions. The matrices  $\mathbf{R}, \mathbf{A}$  are invertible matrices of dimension  $\text{rank}(\hat{V}^\infty \cap \check{V}^\infty)$ .

*Proof.* We shall use the following simple observation to choose the functions  $\mathbf{I}$  and  $\mathbf{J}$  – if  $K = \{\omega^1, \omega^2, \dots, \omega^k\}$  is any completely integrable Pfaffian (where the  $\omega^i$  are independent 1-forms), then there are (locally defined) functions whose differentials will complete any given subset of the generators  $\{\omega^{i_1}, \dots, \omega^{i_p}\}$  to a basis of  $K$ .

Accordingly, choose independent functions  $\mathbf{I}_3 \in \text{Inv}(\hat{V}^\infty)$  such that  $\hat{V}^\infty = \text{span}\{\boldsymbol{\tau}, \hat{\boldsymbol{\eta}}, d\mathbf{I}_3\}$  and let  $\hat{\boldsymbol{\sigma}}_0 = d\mathbf{I}_3$ . Then choose independent functions  $\mathbf{I}_2 \in \text{Inv}(\hat{V}^\infty)$  such that  $\hat{V}^\infty = \text{span}\{\boldsymbol{\tau}, d\mathbf{I}_2, \hat{\boldsymbol{\sigma}}_0\}$ . The 1-forms  $\hat{\boldsymbol{\eta}}$  can therefore be written as

$$\hat{\boldsymbol{\eta}} = \mathbf{P}d\mathbf{I}_2 + \mathbf{G}\boldsymbol{\tau} + \mathbf{H}\hat{\boldsymbol{\sigma}}_0,$$

where  $\mathbf{P}$  is an invertible matrix of functions. Let

$$\hat{\boldsymbol{\eta}}_0 = \mathbf{P}^{-1}(\hat{\boldsymbol{\eta}} - \mathbf{G}\boldsymbol{\tau}) = d\mathbf{I}_2 + \mathbf{E}\hat{\boldsymbol{\sigma}}$$

and note (by (C)) that  $\hat{\boldsymbol{\eta}}_0 \in \hat{V}^\infty \cap \check{V}$  and  $\hat{V}^\infty = \text{span}\{\boldsymbol{\tau}, \hat{\boldsymbol{\eta}}_0, \hat{\boldsymbol{\sigma}}_0\}$ . Finally, if we choose  $\mathbf{I}_1 \in \text{Inv}(\hat{V}^\infty)$  such that  $\hat{V}^\infty = \text{span}\{d\mathbf{I}_1, \hat{\boldsymbol{\eta}}_0, \hat{\boldsymbol{\sigma}}_0\}$  then the first equation in (C.7) holds. Similar arguments allow us to choose  $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$  so that (C.6) and (C.7) hold.  $\blacksquare$

**Remark C.4.** The number of invariants  $\mathbf{I}_1$  and  $\mathbf{J}_1$  are the same and equals the rank of the bundle  $\hat{V}^\infty \cap \check{V}^\infty$ .

**Lemma C.5.** For the co-frame  $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}_0, \check{\boldsymbol{\sigma}}_0, \hat{\boldsymbol{\eta}}_0, \check{\boldsymbol{\eta}}_0, \boldsymbol{\tau}\}$  we have structure equations

$$d\hat{\boldsymbol{\eta}}_0 = \mathbf{a}\boldsymbol{\tau} \wedge \hat{\boldsymbol{\sigma}}_0 + \mathbf{b}\hat{\boldsymbol{\eta}}_0 \wedge \hat{\boldsymbol{\sigma}}_0 + \mathbf{c}\hat{\boldsymbol{\sigma}}_0 \wedge \hat{\boldsymbol{\sigma}}_0. \quad (\text{C.8})$$

*Proof.* Lemma C.3 gives

$$d\hat{\boldsymbol{\eta}}_0 = d\mathbf{E} \wedge \hat{\boldsymbol{\sigma}}_0.$$

But, by definition, the 1-forms  $\hat{\boldsymbol{\eta}}_0$  belong to  $\mathcal{I}^1$  and therefore, by the decomposability condition (4.5), the 1-forms  $d\mathbf{E}$  contain no  $\check{\boldsymbol{\sigma}}_0$  terms. Lemma C.2 then gives  $d\mathbf{E} = \mathbf{a}\boldsymbol{\tau} + \mathbf{b}\hat{\boldsymbol{\eta}}_0 + \mathbf{c}\hat{\boldsymbol{\sigma}}_0$  and (C.8) holds.  $\blacksquare$

*Proof of Theorem 4.3.* Equation (C.2) holds for any 0-adapted co-frame, in particular it holds for the 0-adapted coframe  $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}_0, \check{\boldsymbol{\sigma}}_0, \hat{\boldsymbol{\eta}}_0, \check{\boldsymbol{\eta}}_0, \boldsymbol{\tau}\}$  constructed in Lemma C.3. We now compare equation (C.2) with the result of taking the exterior derivative of equation (C.7) and utilizing (C.6) and (C.8),

$$\begin{aligned} d\boldsymbol{\tau} &= (d\mathbf{R})\mathbf{R}^{-1} \wedge \boldsymbol{\tau} + \mathbf{R}(d\mathbf{S} \wedge \hat{\boldsymbol{\eta}}_0 + \mathbf{S}d\hat{\boldsymbol{\eta}}_0 + d\mathbf{T} \wedge \hat{\boldsymbol{\sigma}}_0) \\ &= (d\mathbf{R})\mathbf{R}^{-1} \wedge \boldsymbol{\tau} + \mathbf{R}(d\mathbf{S} \wedge \hat{\boldsymbol{\eta}}_0 + \mathbf{S}(\mathbf{a}\boldsymbol{\tau} \wedge \hat{\boldsymbol{\sigma}}_0 + \mathbf{b}\hat{\boldsymbol{\eta}}_0 \wedge \hat{\boldsymbol{\sigma}}_0 + \mathbf{c}\hat{\boldsymbol{\sigma}}_0 \wedge \hat{\boldsymbol{\sigma}}_0) + d\mathbf{T} \wedge \hat{\boldsymbol{\sigma}}_0). \end{aligned} \quad (\text{C.9})$$

Now decomposability implies  $d\mathbf{T}$  is independent of the forms  $\check{\sigma}_0$ . By Lemma C.2,  $d\mathbf{T} \in \mathcal{S}(\hat{V}^\infty) = \{\hat{\eta}_0, \hat{\sigma}_0, \tau\}$ . Hence, from (C.9), we deduce that there are no terms of the form  $\check{\eta}_0 \wedge \hat{\sigma}_0$  in  $d\tau$ . Therefore  $\mathbf{a}_3 = 0$  in equation (C.2) (in the current 0-adapted coframe). A similar argument using the expression for  $\tau$  in terms of the invariants  $\mathbf{J}$  gives  $\mathbf{a}_2 = 0$  in equation (C.2), leaving

$$d\tau = \alpha \wedge \tau + \mathbf{a}_1 \check{\eta}_0 \wedge \hat{\eta}_0. \quad (\text{C.10})$$

Returning to equation (C.9), we now conclude, from the absence of the terms  $\hat{\eta}_0 \wedge \check{\sigma}_0$  in (C.10), that  $d\mathbf{S}$  is free of  $\check{\sigma}_0$  terms and so, again by Lemma C.2,  $d\mathbf{S} \in \mathcal{S}(\hat{V}^\infty)$ . Consequently equation (C.9) implies that  $d\tau$  contains no terms  $\hat{\eta}_0 \wedge \check{\eta}_0$  and therefore  $\mathbf{a}_1 = 0$  in equation (C.10). This proves that  $\hat{V}^\infty \cap \check{V}^\infty$  is an integrable Pfaffian system which, by the remarks following the statement of Theorem 4.3, shows that  $\hat{V}^\infty \cap \check{V}^\infty = \{0\}$ .  $\blacksquare$

## D Group Actions

In this appendix we gather together the details of various facts about group actions used in the paper. We begin with Remark 3.3.

**Theorem D.1.** *Let  $G$  act freely and regularly on  $M$  and let  $H \subset G$  be a closed subgroup. Then  $H$  acts regularly on  $M$ .*

*Proof.* Let  $p_1, p_2 \in M$  and  $[p_1] = \mathbf{q}_N(p_1), [p_2] = \mathbf{q}_N(p_2) \in M/N$ . We show  $M/H$  is Hausdorff. If  $\mathbf{p}([p_1]) \neq \mathbf{p}([p_2])$  then it is easy to separate  $[p_1]$  and  $[p_2]$  using separating open sets in  $M/G$ . So supposed we are in the other situation in which case  $p_2 = a \cdot p_1$ , but  $p_2 \notin H \cdot p_1$ . Let  $\Phi : U \rightarrow \bar{U} \times G$  be a trivialization, then  $\Phi(p_1) = (\bar{p}, a)$  and  $\Phi(p_2) = (\bar{p}, b)$  and  $ab^{-1} \notin H$ . Since  $H$  is closed in  $G$ , we may separate  $a$  and  $b$  by  $H$ -invariant open sets  $A, B$  in  $G$ . Then  $U_A = \Phi^{-1}(\bar{U} \times A)$  and  $U_B = \Phi^{-1}(\bar{U} \times B)$  are  $H$ -invariant open sets in  $M$  which don't intersect. Their images  $\mathbf{q}_H(U_A)$  and  $\mathbf{q}_H(U_B)$  separate  $[p_1]$  and  $[p_2]$ .

In a similar manner using  $\pi : G \rightarrow G/H$  the composition  $(I, \pi) \circ \Phi : U \rightarrow \bar{U} \times G/H$  gives rise to a map  $\tilde{\Phi} : \mathbf{q}_H(U) \rightarrow \bar{U} \times G/H$  which in turn defines a differentiable structure on  $M/H$ .  $\blacksquare$

In Section 8, we use the following. See also exercise 9 page 134 of [37].

**Proposition D.2.** *Let  $G, H$  be Lie groups with  $H$  connected. Let  $\phi : H \rightarrow G$  be a smooth map satisfying  $\psi(e_h) = e_G$  and  $\psi_* : \mathfrak{h} \rightarrow \mathfrak{g}$  a Lie algebra monomorphism. Then  $\phi$  is a homomorphism.*

*Proof.* Let  $X$  be a right invariant vector-field on  $H$  and  $Y$  the right invariant vector-field on  $G$  satisfying  $\phi_*(X) = Y$ . We claim

$$\phi(\exp(tX)a) = \exp(tY)\phi(a) \quad \text{for all } t \in \mathbf{R}, \quad a \in H. \quad (\text{D.1})$$

The curve  $\exp(tX)a$  is the maximal integral curve of  $X$  through  $a$  and  $\exp(tY)\phi(a)$  is the maximal integral curve of  $Y$  through  $\phi(a)$ . The hypothesis  $\phi_*X = Y$  gives

$$\phi_*\left(\frac{d}{dt}\exp(tX)a\right) = Y|_{\phi(\exp(tX)a)}$$

and  $\phi(\exp(0X)a) = \phi(a)$ . Therefore  $\phi(\exp(tX)a)$  is also the maximal integral curve of  $Y$  through  $\phi(a)$ , and so equation (D.1) holds.

We can now show  $\phi$  is a homomorphism. Let  $a, b \in H$ , and by the connectivity of  $H$  write  $a = \Pi_{i=1}^k \exp(t_i X_i)$ ,  $X_i \in \mathfrak{h}$ . Then by (D.1), induction and the fact  $\phi(e_h) = e_g$  we have

$$\phi(a) = \phi(\Pi_{i=1}^k \exp(t_i X_i)) = \Pi_{i=1}^k \exp(t_i Y_i) \phi(e_H) = \Pi_{i=1}^k \exp(t_i Y_i)$$

where  $Y_i \in \mathfrak{g}$  with  $\phi_* X_i = Y_i$ . This implies again by equation (D.1) that

$$\phi(ab) = \phi(\Pi_{i=1}^k \exp(t_i X_i)b) = \Pi_{i=1}^k \exp(t_i Y_i) \phi(b) = \phi(a)\phi(b)$$

and  $\phi$  is a homomorphism. ■

**Theorem D.3.** *Let  $H$  be a connected Lie group acting freely and regularly on  $N$  and let  $G$  be a Lie group acting freely and regularly on  $M$ . Suppose  $\Phi : N \rightarrow M$  satisfies*

$$\Phi_* : \Gamma_H \rightarrow \Gamma_G \tag{D.2}$$

*and is a monomorphism of the Lie algebra of infinitesimal generators. Then there exist a homomorphism  $\phi : H \rightarrow G$  such that*

$$\Phi(hp) = \phi(h)\Phi(p).$$

*That is  $\Phi$  is  $H$  equivariant.*

*Proof.* Let  $p \in N$  then by the regularity hypothesis, the map  $\iota : H \rightarrow M$  given by

$$\iota_p(h) = hp.$$

is an embedding. Equation (D.2) implies  $\Phi \circ \iota : H \rightarrow M$  satisfies  $(\Phi \circ \iota)_* TH \subset \Gamma_G$  and  $\Phi(\iota(e)) = \Phi(p)$ . Therefore  $\Phi(\iota(H))$  is contained in the maximal integral manifold of  $\Gamma_G$  through  $\Phi(p)$ , and hence

$$\Phi(\iota(H)) \subset O_G(\Phi(p)). \tag{D.3}$$

Since  $G$  acts regularly, the map  $\tau : G \rightarrow O_G(\Phi(p))$  given by  $\tau(g) = g\Phi(p)$  is also an embedding. Therefore  $\Phi \circ \iota$  induces a smooth map  $\phi : H \rightarrow G$ . In particular by the freeness of the action of  $G$ , there exists a unique  $g \in G$  such that

$$\Phi(hp) = \tau(g) = g\Phi(p) \tag{D.4}$$

and by definition  $\phi(h) = g$ . Equation (D.4) then reads

$$\Phi(hp) = \phi(h)\Phi(p). \quad (\text{D.5})$$

Let  $h$  be the Lie algebra of right invariant vector-fields on  $H$ , then  $\iota_* : h \rightarrow \Gamma_H$  is a Lie algebra isomorphism. Furthermore  $\Phi_*\iota_* : h \rightarrow \Gamma_G$ , is a monomorphism, and likewise so is  $\phi_*h \rightarrow \mathfrak{g}$ . By Proposition D.2 is a homomorphism.

The homomorphism  $\phi$  can be constructed for each  $p \in N$ , however by the uniqueness theorem for homomorphisms (Theorem 3.16 page 92 Warner),  $\phi$  does not depend on  $p$ . Therefore equation (D.5) holds for all  $\in N$ . ■

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